A Discrete Approach to Continuous Logistic Growth

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Outline

- Continuous Logistic Growth
- Discrete Logistic Growth: Linear Growth Factor
- Alternate Approach: Inverse Linear Growth Factor
- Slides and a detailed paper available at website

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Continuous Logistic Growth

- Covered in many prior-to-calc courses that emphasize modelling
- Introduced in context of limited growth
- Family of functions approach:

$$p(t) = \frac{A}{1+Be^{-kt}} \quad \text{or} \quad p(t) = \frac{A}{1+Bb^{t}}$$

- Usually no rationale for functions of this type

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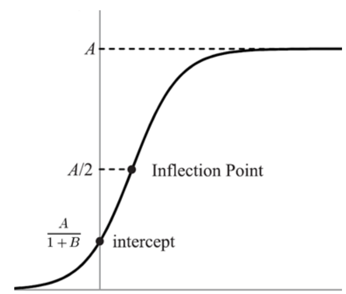

Symmetry about (t^*, $A/2$) provable by elementary methods:

- $Bb^{t^*} = 1$
- Show $f(t^*+a) - A/2 = A/2 - f(t^*- a)$
- This shows that (t^*, $A/2$) is an inflection point

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Applied Problems

- Made up problems: give a few points and solve for parameters
- Special case: postulate the location of the inflection point and the intercept
- Fitting data: estimate inflection point and intercept from graph
- Least Squares fit using built in routines on a calculator (TI 83Plus) or …

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Pedagogical Pros and Cons

PRO

- Nice equation, graph, analysis methods
- Realistic modelling examples
- Widely used

CON

- Difficult to Motivate w/o calculus
- Typical quote: "*Deriving the formula for logistic growth requires techniques beyond the scope of this text, and we simply state the result.*" [Crauder, Evans, and Noell, *Functions and Change*: *A Modeling Approach to College Algebra*, 5E, Cengage p 319.]

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Discrete Logistic Growth

- Discrete model: sequence $\{p_n\}$
- Difference equation: $p_{n+1} = f(p_n)$
- Exponential Growth: $p_{n+1} = r \cdot p_n$, constant r, unsustainable
- Refined model: $p_{n+1} = r(p_n) \cdot p_n$, growth factor decreases as population increases.
- "Simple" case: r decreases linearly with p:
  $p_{n+1} = (-mp_n + b) \cdot p_n = m(\mathrm{L} - p_n) \cdot p_n$
- This is commonly called the discrete logistic model in texts and on webpages (refs in ms)

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Discrete Model Example

- Initial population = 1000
- Initial growth factor = 1.4
- Equilibrium population = 5000
- Linear equation: $r\,(1000) = 1.4$, $r\,(5000) = 1$
- $r\,(p) = 1.5 - .0001p$
- Population model:
  $p_{n+1} = r(p_n)\, p_n = (1.5 - .0001p_n)\, p_n$

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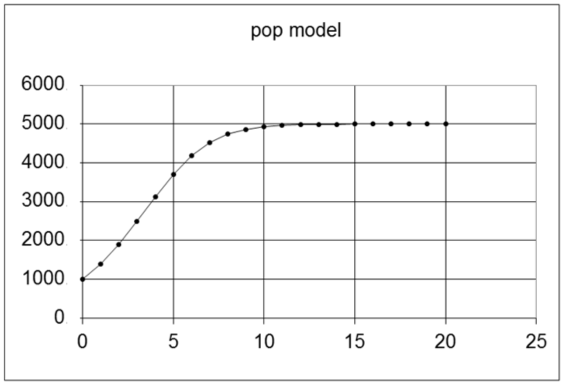
n	p_n
0	1000.0
1	1400.0
2	1904.0
3	2493.5
4	3118.5
5	3705.2
6	4185.0
7	4526.1
8	4740.6
9	4863.6
10	4929.9
11	4964.5
12	4982.1
13	4991.0
14	4995.5
15	4997.7
16	4998.9
17	4999.4
18	4999.7
19	4999.9
20	4999.9



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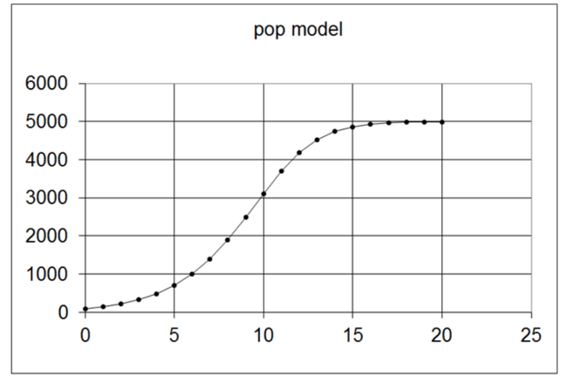
n	p_n
0	100.0
1	149.0
2	221.3
3	327.0
4	479.8
5	696.7
6	996.6
7	1395.5
8	1898.5
9	2487.4
10	3112.3
11	3699.9
12	4180.9
13	4523.3
14	4739.0
15	4862.7
16	4929.4
17	4964.2
18	4982.0
19	4991.0
20	4995.5



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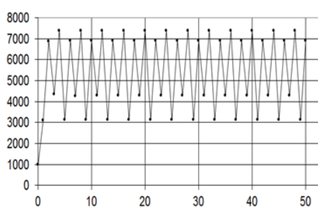
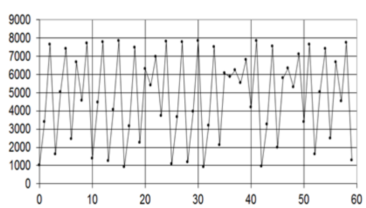
Limitations

- No simple equation for p_n as a function of n.
- The *nice* behavior of the example doesn't always occur
- Depending on parameter values, models can oscillate, or exhibit chaotic behavior





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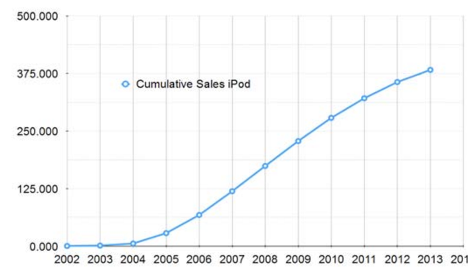
Cumulative Ipod Sales



- Looks like a candidate for a logistic model
- Structurally, we expect cumulative sales to exhibit limited growth
- Discrete logistic growth model poor fit
- Continuous logistic growth fits pretty well

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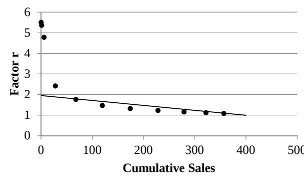
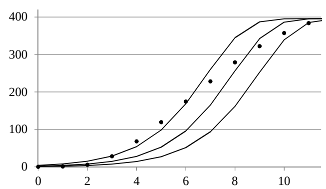
First Discrete Fit Attempt



- Note nonlinear *r* data
- One possible linear fit shown
- Gives us coefficients for difference equation



- Here are some results with different initial values
- None of models fit the data well

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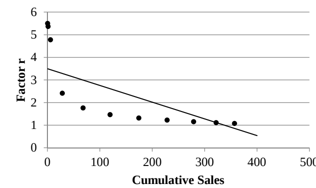
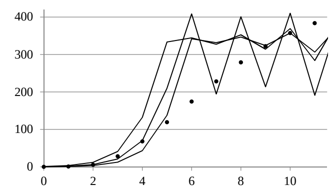
Second Discrete Fit Attempt



- This linear fit for *r* factor tries to better match initial observations



- Three initial values shown
- Again, none of models fit the data well
- Note the oscillation in the model

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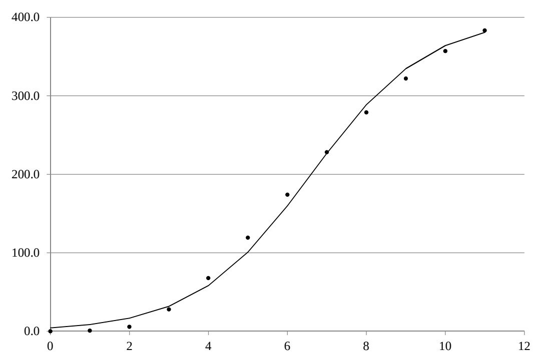
Continuous Logistic Growth Model Fit to Ipod Data



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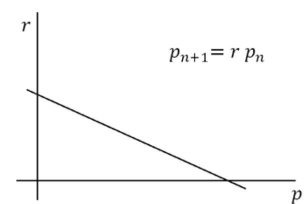
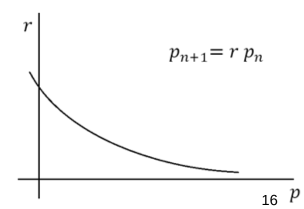
Rationale for a Refined Discrete Model

- Is linear model for growth factor reasonable?



- What else could we try?
- How about inverse linear:

$$r(p) = \frac{1}{mp + b}$$



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Example 1 Revisited

- Initially $p_0 = 1000$, $r = 1.4$
- Equilibrium population = 5000
- $r\,(1000) = 1.4$, $r\,(5000) = 1$
- Values for $1/r$ **:** (1000, 1/1.4), (5000, 1)
- Linear Equation: $\frac{1}{r} = \frac{1}{14000}p + \frac{9}{14} = \frac{p+9000}{14000}$
- $r(p) = \frac{14000}{p+9}$
- Population model:

$$p_{n+1} = r(p_n)\,p_n = \frac{14000p_n}{p_n+9000}$$

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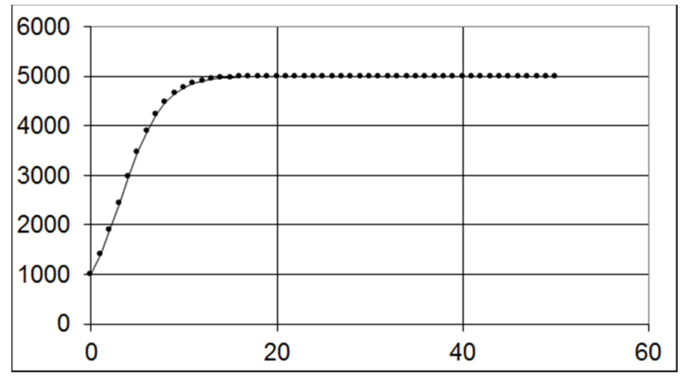

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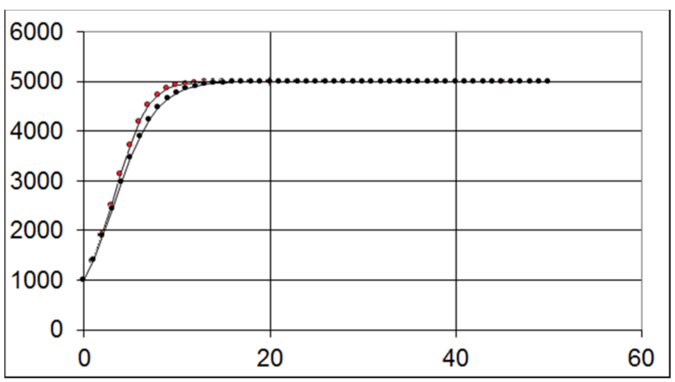

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Explicit Equation for the Refined Model

- $p_{n+1} = \frac{p_n}{mp_n+b} \Rightarrow \frac{1}{p_{n+1}} = \frac{mp_n+b}{p_n} = m + b\,\frac{1}{p_n}$
- This is a linear diff equation in $1/p_n$
- Solution is a shifted exponential

$$\frac{1}{p_n} = \alpha + \beta b^n$$

- $p_n = \frac{1}{\alpha+\beta b^n} = \frac{\frac{1}{\alpha}}{1+\frac{\beta}{\alpha}b^n} = \frac{A}{1+Bb^n}$
- Solution to refined model is given by a continuous logistic growth equation

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Summary

- Natural progression of discrete models leads to continuous logistic growth
  - Exponential Growth
  - *Standard* Discrete Logistic Growth
  - *Refined* Discrete Logistic Growth
- Development reveals important aspects of modeling
  - Postulation / Analysis / Refinement cycle
- Provides a motivated derivation of the continuous logistic growth model

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