Worksheet 2: Permutations and Symmetries

Let us assign labels to the following elements of S_3:

$$e = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}; \quad \rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}; \quad \varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}$$

The composition $\rho\circ\phi$ is defined as the effect of first applying ϕ and then applying ρ to the result. Thus, since $\phi(1) = 2$ and $\rho(2) = 3$, we see that $\rho\circ\phi(1) = \rho(\phi(1)) = \rho(2) = 3$.

1. Find the following: $\rho\circ\phi(2) =$ ____________ $\rho\circ\phi(3) =$ ____________

   $\rho\circ\phi(4) =$ ____________ $\rho\circ\phi(5) =$ ____________ $\rho\circ\phi(6) =$ ____________

2. Use the answers to 1 to complete the following representation of $\rho\circ\phi$

$$\rho \circ \phi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & \square & \square & \square & \square & \square \end{pmatrix}$$

3. Using similar methods find the following:

   a. $\rho\circ\rho =$ ________________________ d. $\rho\circ\rho\circ\rho =$ ______________________

   b. $\phi\circ\phi =$ ________________________ e. $\rho\circ\rho =$ ________________________

   c. $\phi\circ\rho =$ ________________________ f. $\phi\circ\rho\circ\rho =$ ________________________

From now on we will leave out the $\circ$ symbol, and just write $\phi\rho$ instead of $\phi\circ\rho$. Similarly, we'll write ρ^2 for $\rho\circ\rho$, ρ^3 for $\rho\circ\rho\circ\rho$, etc.

4. Using the results of 3, observe that the following 6 permutations are all distinct: e, ρ, ρ^2, ϕ, $\phi\rho$, $\phi\rho^2$. Since there are only 6 possible permutations of {1,2,3}, the group S_3 must be given by $S_3 = \{e, \rho, \rho^2, \phi, \phi\rho, \phi\rho^2\}$. Show that $\rho\phi = \phi\rho^2$ and $\rho^2\phi = \phi\rho$.

5. Complete a multiplication table for $S_3 = \{e, \rho, \rho^2, \phi, \phi\rho, \phi\rho^2\}$. Each entry in the body of the table should be one of the six elements e, ρ, ρ^2, ϕ, $\phi\rho$, and $\phi\rho^2\}$. To simplify products like $(\rho^2)(\phi\rho)$ use the following facts: $\phi^2 = \rho^3 = e$, $\rho\phi = \phi\rho^2$ and $\rho^2\phi = \phi\rho$. For example $(\rho^2)(\phi\rho) = (\rho^2\phi)(\rho) = (\phi\rho)(\rho) = \phi\rho^2$.

	e	ρ	ρ^2	ϕ	$\phi\rho$	$\phi\rho^2$
e						
ρ						
ρ^2					$\phi\rho^2$	
ϕ						
$\phi\rho$						
$\phi\rho^2$						

Symmetries of an equilateral triangle. We recognize that a figure has left-right symmetry if each side is a mirror image of the other. Another way to say the same thing is this: If you draw the trace the figure on a clear plastic sheet, and then flip that sheet over, it will exactly overlay on the original. This second idea has been adopted in mathematics as the defining characteristic of symmetry. We recognize certain transformations of plane figures as leaving the *shapes* unchanged. These include rotating a figure by some angle about a fixed point, sliding the figure a certain distance in a specified direction, and reflecting a figure across a specified line. Any such transformation that leaves a figure indistinguishable from its original configuration is considered to be a symmetry of the figure. For example, rotating an equilateral triangle by 120° about its center point is a symmetry. Rotating it by an equal amount about its top vertex is not a symmetry – you can see that the rotated figure is not identical to the original, because it's location is different. All of the symmetries of an equilateral triangle are either rotations or reflections.

6. Find all of the rotational symmetries of the equilateral triangle. For each specify the center point of the rotation and the angle of rotation. Consider two rotations R and S to be identical if for every point p of the triangle, $R(p) = S(p)$. Thus for example, a clockwise rotation about the top vertex by 90° is identical to a counter-clockwise rotation about the top vertex by 270°.

7. Find all of the reflection symmetries of the equilateral triangle. For each specify the location of the line across which the reflection acts.

8. The set of symmetries of a figure is always a group with the operation being composition. How many elements did you find in the group of symmetries of the

   equilateral triangle? ____________

9. Pick a pair of symmetries, call them f and g. Describe each below. Then describe the compositions fg and gf. Are they both previously identified symmetries of the triangle? Is $fg = gf$? [Hint: one way to keep track of compositions is to label the vertices of the triangle, say A, B, and C. You can compare different combinations of symmetries by comparing where the labels end up. If the labels end up in identical locations after applying fg as they do after applying gf, then $fg = gf$. Similarly, if the lables look the same after applying fg as they do after applying h, then $fg = h$.]

   f = ____________________________ g = ______________________________

   fg = ___________________________ gf = ______________________________