Extra Problems Section 4.3 Solutions in Red

1. Solve the initial value problem

$$y'' + 4y = \cos \alpha t\,; \quad y(0) = y'(0) = 0,$$

where α is a real constant.

[Solution: $y(t) = \dfrac{1}{\alpha^2 - 4}\left(\cos(2t) - \cos(\alpha t)\right)$.]

2. Continuing the prior problem, suppose that $\alpha = 2 + \varepsilon$. Use the trigonometric identity $\cos(A) - \cos(B) = 2\sin\left(\dfrac{B-A}{2}\right)\sin\left(\dfrac{B+A}{2}\right)$ to show that the solution to the initial value problem in 1 can be expressed as

$$y(t) = \frac{2}{\varepsilon(4+\varepsilon)}\left(\sin\left(\frac{\varepsilon}{2}t\right)\sin\left(\left(2+\frac{\varepsilon}{2}\right)t\right)\right).$$

Solution: Replace A with $2t$ and B with αt in the trig identity. That leads to $\cos(2t) - \cos(\alpha t) = 2\sin\left(\dfrac{\alpha-2}{2}t\right)\sin\left(\dfrac{\alpha+2}{2}t\right)$. But notice that $\alpha - 2 = \varepsilon$ and $\alpha + 2 = 4 + \varepsilon$.

Thus we have $\cos(2t) - \cos(\alpha t) = 2\sin\left(\dfrac{\varepsilon}{2}t\right)\sin\left(\dfrac{4+\varepsilon}{2}t\right) = 2\sin\left(\dfrac{\varepsilon}{2}t\right)\sin\left(\left(2+\dfrac{\varepsilon}{2}\right)t\right)$. Finally, multiply both sides by $\frac{1}{\alpha^2-4} = \frac{1}{(\alpha-2)(\alpha+2)} = \frac{1}{\varepsilon(4+\varepsilon)}$ to complete the derivation.

3. Use the result of 2 to analyze the graph of $y(t)$ when $\varepsilon = .2$. What is the numerical value of the coefficient $\dfrac{2}{\varepsilon(4+\varepsilon)}$? Answer: 2/(.2·4.2) = 2/.84 = 2.38. What is the period of the first sine factor? Answer: period = 4π/ε = 4π/.2 = 20π. The second sine factor? Answer: period = 4π/(4+ε) = 4π/(4.2) = .95π approximately. Using these results, describe the graph.

Answer: The longer wave length oscillation has amplitude 2.38 and repeats every 20π units. This curve, and its reflection across the x axis creates an envelope within which the actual solution curve must stay. The solution curve oscillates between the top and bottom of the envelope, completing a complete cycle in a little less than π units. This means that a little more than 10 complete cycles occur within the envelope of a single cycle of the slower oscillation.

4. Repeat problem 3 using first ε = .02 then ε = .002. What do these results suggest about the behavior of the system as $\alpha \to 2$?

Solution: See on the next page a table showing the amplitude and periods of both oscillations for the two specified values of ε.

ε	Amplitude $= \dfrac{2}{\varepsilon(4+\varepsilon)}$	Period of first factor = 4π/ε	Period of second factor = 4π/(4+ε)
.02	24.87…	200π	.995π
.002	249.87…	2000π	.9995π

These results suggest that as ε decreases toward 0, the solution curve approaches an oscillation of period approximately p, taking place within a much slower, much taller oscillation, with both amplitude and period going to infinity.

5. How do parts 3 and 4 relate to the concept of resonance?

Answer: At resonance the oscillation takes place with period π, and with amplitude that increases linearly with time. That is, the envelope for the oscillation is a pair of lines through the origin making equal angles with the *x* axis. A system that is near resonance, the oscillation will approximate the resonant behavior at first, but eventually the amplitude with round out and diminish back toward zero, settling into a repetitive pattern of increasing and then decreasing amplitudes. So we can see that as the system approaches resonance, the solution curves do approach the resonant solution curve.