Differential Equations First Exam Review

The first exam will cover chapter 1 sections 1 through 8. You will be expected to have a conceptual understanding of the important ideas we have discussed, as well as the mathematically correct statements of specific theorems and principles. This includes various types of graphs, and you should be able to explain what they are and how they are interpreted. You should be able to define and correctly use notation and terminology. In addition, you should know and be able to correctly use the procedures that we have covered. These things are outlined below. The outline includes some specific homework exercises. There is also a list of review exercises on the assignments page, covering many of the concepts and procedures you should know.

Concepts

1. Differential equation; solution to a differential equation; initial value problem; solution to an initial value problem vs solution to a differential equation
2. Development of a differential equation as part of a model of some phenomenon or system, including the cycle of formulating a differential equation, studying the solutions, comparing the model with the actual phenomenon or system, and then refining the differential equation.
3. Numerical, Qualitative, and Analytic methods
4. Equilibrium Solutions
5. Types of differential equations: autonomous, linear, separable, first order
6. Slope field for a differential equation $y' = f(t, y)$; special properties when f depends only on t or only on y
7. Euler's Method to approximate solutions to a differential equation; propagation of errors; significance of step size
8. Existence and Uniqueness Theorems
9. Solution curves that go to infinity in finite time; give an example (at least in the form of a sketch) (see page 65); explain the practical significance in a model
10. Differential equations where uniqueness does not hold; give an example (at least in the form of a sketch) (see page 67); identify differential equations where uniqueness cannot be assumed to hold at a particular point.
11. Classification of equilibria as sources, sinks, and nodes
12. Differential equations depending on a parameter; notation for such equations
13. bifurcation; bifurcation value or point
14. linearity of a differential equation
15. homogeneous and nonhomogeneous equations, *forced* and *unforced* terminology.

Graphs

Be able to explain, create, and interpret the following types of graphs

1. Solution curves showing $y(t)$ vs t.
2. Slope fields
3. Phase lines
4. Graphs of $f(y)$ vs y for a differential equation $y' = f(y)$
5. Bifurcation Diagram

Standard Equations

Know the following standard forms for various types of differential equations

1. First order differential equation $y' = f(t, y)$
2. Autonomous (first order) differential equation $y' = f(y)$
3. Logistic population model $y' = k(1 - y/N)y$ or $P' = k(1 - P/N)P$
4. Separable (first order) differential equation $y' = g(t)h(y)$
5. First order linear equation $y' = a(t)y + b(t)$, homogeneous case $y' = a(t)y$

Theorems and Principles

1. For autonomous differential equations horizontally shifting a solution curve produces another solution curve.
2. For a differential equation of the form $y' = f(t)$, shifting a solution curve vertically produces another solution curve.
3. Existence Theorem
4. Uniqueness Theorem
5. Extended version of existence and uniqueness theorem (not in the text): Suppose $f(t, y)$ and $\partial f/\partial y$ are continuous functions on a rectangle of the form $R = \{(t, y) \mid a < t < b, c < y < d\}$ in the ty-plane. If $(t_0, y_0) \in R$ then the initial value problem

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0$$

has a unique solution. Moreover, this solution extends from the point (t_0, y_0) to the boundary of the rectangle R

6. Implications of Existence and Uniqueness: solution curves cannot intersect; solution curves between equilibrium solutions are bounded by those solutions; for autonomous equations, a solution curve between equilibrium solutions must extend to $t = \infty$ and $t = -\infty$ and must be asymptotic to the nearest equilibrium solutions.
7. Linearization Theorem
8. RE the bifurcation diagram for a one parameter family of differential equations of the form $y' = f_\mu(y)$: the curve or curves traced out by equilibrium solutions as μ varies are given by the equation $f_\mu(y) = 0$. That's an equation in the variables μ and y, and it is graphed in a plane with μ on the horizontal axis and y on the vertical.
9. For a one parameter family of differential equations $y' = f_\mu(y)$ a necessary condition for a bifurcation to occur at $\mu = \mu_0$ is the existence of a corresponding y_0 such that

$$\begin{aligned} f_{\mu_0}(y_0) &= 0 \quad \text{and} \\ f'_{\mu_0}(y_0) &= 0. \end{aligned}$$

   This doesn't mean that every such point (μ_0, y_0) is a bifurcation point (just like every point where the derivative of a function is zero doesn't have to be a max or a min). But it does mean, if you find every possible point where the two equations above hold, that these are the only possible bifurcation points. Once you find them, you have to do further analysis (for example, checking what happens to the phase line at such a point when the parameter changes) to decide if it really is a bifurcation point.
10. Continuing the previous two items, suppose that (μ_0, y_0) is a point in the $\mu - y$ plane. The equation $f_{\mu_0}(y_0) = 0$ says that (μ_0, y_0) is on one of the curves in the bifurcation diagram, and the equation $f'_\mu(y) = 0$ says that the tangent line at that point of the curve is vertical. This can be proved with methods from calculus 3 as follows. Rewrite the equation for the curve in the form $G(\mu, y) = 0$. At any point of the curve, the gradiant vector $\nabla G = (\partial G/\partial \mu)\mathbf{i} + (\partial G/\partial y)\mathbf{j}$ is perpendicular to the curve. This perpendicular vector will be horizontal (and so the tangent will be vertical) exactly when $\partial G/\partial y = 0$. But $\partial G/\partial y$ is the same thing as $f'_\mu(y)$.
11. Linearity Principle: for a homogeneous linear differential equation linear combinations of solutions are also solutions. In the case of a first order linear equation, this just means that you can multiply solutions by constants and they remain solutions. For higher order linear equations, you can add up solutions to the homogeneous equation and you still have a solution. This is referred to in physics as a superposition principle.
12. Extended Linearity Principle 1: for a linear non-homogeneous differential equation, you can add any solution of the equation to a solution of the corresponding homogeneous equation, and the result is still a solution of the original equation.
13. Extended Linearity Principle 2: for a linear non-homogeneous differential equation, subtracting any two solutions of the equation produces a solution to the corresponding homogeneous equation. In other words, the difference between solutions to the non-homogeneous equation must be a solution of the homogeneous equation.
14. The trivial equilibrium function $y = 0$ is always a solution to a linear homogeneous differential equation.
15. To find the general solution for a non-homogeneous linear differential equation, you can first find the general solution to the corresponding homogeneous equation, then find any single solution to the original equation, and add the two together.

Procedures

1. Find equilibrium solutions for a differential equation
2. Given a differential equation $y' = f(t, y)$, determine where y' is positive, negative, zero, greatest, least, and interpret
3. Modify a given differential equation according to a specified condition (see prob 1.1.17)
4. Given a description of a system or phenomenon, explain why you should expect a model that is or is not autonomous; or give an example of a system that should have an autonomous or a non-autonomous differential equation model
5. Check whether a given function is a solution to a given differential equation and/or initial value problem
6. Use separation of variables method to find general solutions to differential equations as well as solutions to initial value problems; identify *missing solutions* that do not arise from separation of variables
7. sketch a slope field for a differential equation; sketch a solution curve given a particular slope field
8. Perform a qualitative analysis for an autonomous differential equation, and represent the results in the form of a phase line
9. Given a phase line, describe the qualitative analysis it implies
10. Given a graph representing $f(y)$. Sketch the slope field, perform a qualitative analysis, and draw a phase line for the differential equation $y' = f(y)$. See exercises 1.6.29-32.
11. Apply Euler's method to approximate a solution curve to a differential equation

12. Determine whether uniqueness can be assumed to hold for a particular initial value problem (see problem 1.5.11)
13. Use known solutions to a differential equation to draw conclusions about other solutions, using the existence and uniqueness theorem (see problem 1.5.2)
14. Given a differential equation, classify equilibrium solutions as sinks, sources, or nodes. Relate to phase line diagram.
15. Use linearization theorem to identify equilibrium solutions as sinks, sources, or nodes
16. Find bifurcation points graphically, given a bifurcation diagram.
17. Find bifurcation points analytically given a specific differential equation $y' = f_\mu(y)$.
18. Describe qualitatively how solutions to a differential equation change at a bifurcation point, given the bifurcation diagram. (Example, in problem 1.7.13a-d, identify any bifurcation points and describe how the qualitative description of solutions to the corresponding differential equation change as the parameter value moves from one side of the bifurcation point to the other).
19. General method for solving a first order linear differential equation: Write the equation in the form $y' + P'(t)y = b(t)$, for a suitable function $P(t)$. Then multply both sides of the equation by $e^{P(t)}$, observing that the result has the form $[e^{P(t)}y]' = e^{P(t)}b(t)$. Integrate both sides, and solve for y. The end result is $y(t) = e^{-P(t)} \int e^{P(t)} b(t)\, dt$.