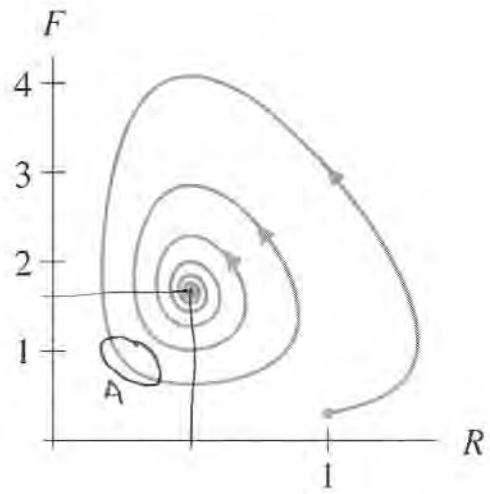
Differential Equations
Fall 2012
Sample Exam 2 Name ______________________

1. The figure at right shows a solution curve for a predator prey model. The R axis represents the size of the prey population (*rabbits*) and the F axis represents the size of the predator population (*foxes*), both in units of 1000's. Based on the graph, answer the following questions, or explain why there is not enough information to do so.



a. In the long term, what will happen to the sizes of the two populations?

They will each approach a constant value, because the solution curve is spiraling toward a point roughly equal to (.5, 1.6)

b. During periods when the rabbit population is increasing, is it also true that the fox population is increasing? Or are there times when rabbits are increasing but foxes decreasing in number?

In the section of the curve marked "A" rabbits are ~~in~~creasing and foxes decreasing.

c. Approximately how long does it take the solution curve to cycle one time around the central point?

Not enough information to tell, because the solution curve does not show information about variable t.

2. Vector Fields and Systems of Differential Equations

A vector field $\mathbf{F}$ and a vector function $\mathbf{Y}$ are defined by

$$\mathbf{F}(x,y) = \begin{bmatrix} xy+2 \\ .5y \end{bmatrix} \quad \text{and} \quad \mathbf{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

a. Express the differential equation $\mathbf{Y}' = F(Y)$ as a system of differential equations

$x'(t) = xy + 2$
$y'(t) = .5y$

b. One solution curve of this system passes through the point (3,2). At that point, what is the slope of the solution curve? Is $x(t)$ increasing or decreasing with time there? Explain.

@ $(3,2)$ $\quad Y' = \begin{bmatrix} 3\cdot 2+2 \\ (.5)2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$ $\quad$ slope $= \frac{1}{8}$. $\quad \frac{dx}{dt} = 8 > 0$

So $x(t)$ is increasing.

c. Find all equilibrium solutions, or explain how you know that there are none.

(x,y) is an equilibrium iff $F(x,y) = 0$

$$\begin{bmatrix} xy+2=0 \\ .5y=0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} xy+2=0 \\ y=0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x\cdot 0+2=0 \\ y=0 \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} 2=0 \\ y=0 \end{bmatrix}$$

There are no solutions b/c $2 \neq 0$ for any $x.y$.

d. For what initial points does the existence and uniqueness theorem apply? Explain.

$F(x,y)$, $\frac{\partial F}{\partial x}$, and $\frac{\partial F}{\partial y}$ are all polynomials in x & y and so all continuous and differentiable everywhere in the (x,y) plane. Therefore the E. & U. theorem applies everywhere (in any rectangles).

3. A mass and spring system has mass of 2.5, a damping coefficient of 10, and a spring constant of 15.

a. Let $y(t)$ represent the location of the mass at time t, and assume that $y = 0$ when the spring is neither stretched nor compressed. Find the differential equation for $y(t)$.

$F = ma = my''$. $F_{spring} = -ky = -15y$ $F_{damping} = -by' = -10y'$

$-15y - 10y' = my'' = 2.5y''$

$2.5y'' + 10y' + 15y = 0$

$y'' + 4y' + 6y = 0$

b. Reformulate this equation as a system of first order equations.

Let $v = y'$. Then $v' = y'' = -4y' - 6y = -4v - 6y = -6y - 4v$

This gives us the system

$y' = v$
$v' = -6y - 4v$ or in matrix form $\begin{bmatrix} y \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -6 & -4 \end{bmatrix} \begin{bmatrix} y \\ v \end{bmatrix}$

c. Does the matrix in your answer to part b have real or nonreal eigenvalues? What does this imply about the motion of the spring from any nonzero starting point? Explain.

To find eigenvalues, solve det $[A - \lambda I] = 0$

$\begin{vmatrix} -\lambda & 1 \\ -6 & -4-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 6 = 0$ solns: $\lambda = \frac{-4 \pm \sqrt{16 - 24}}{2}$

$\lambda = \frac{-4 \pm \sqrt{-8}}{2} = -2 \pm i\sqrt{2}$

the eigenvalues are non-real. This tells us that the solution curve ~~to~~ the system spirals around the origin.

Therefore $y(t)$ oscillates forever.

4. For the system of differential equations

$$x' = 3x + xy \quad (1)$$
$$y' = 72t^3/y \quad (2)$$

find a solution with $(x, y) = (1, 6)$ at $t = 1$.

Solve eqn (2): $\frac{dy}{dt} = \frac{72t^3}{y} \to y\,dy = 72t^3 dt \to \int y\,dy = \int 72t^3 dt$

$\to \frac{y^2}{2} = 18t^4 + C$. Since $y = 6$ @ $t = 1$, $\frac{6^2}{2} = 18(1)^4 + C \to 18 = 18 + C$ so $C = 0$.

So $\frac{y^2}{2} = 18t^4 \to y^2 = 36t^4 \to y = \pm 6t^2$. $y = 6$ at $t = 1$ so $y = 6t^2$.

Substituting in eqn (1) gives

$x' = 3x + xy = x(3+y) = x(3+6t^2)$ Separating variables,

$\frac{1}{x}dx = (3+6t^2)dt \to \int \frac{1}{x}dx = \int (3+6t^2)\,dt \to \ln|x| = 3t + 2t^3 + K$.

Using $x = 1$ at $t = 1$, $\ln|1| = 3 + 2 + k$ so $0 = 5 + k$ and $k = -5$. So $\ln|x| = 3t + 2t^3 - 5$. Exponentiating,

$$e^{\ln|x|} = e^{3t+2t^3-5}$$

so

$$|x| = e^{3t+2t^3-5}$$

and since $x = 1 > 0$ @ $t = 1$, $|x| = x$. This makes our solution $(x(t), y(t)) = (e^{3t+2t^3-5}, 6t^2)$.

Check: at $t = 1$, $(x(1), y(1)) = (e^{3+2-5}, 6) = (1, 6)$ ✓

$x' = (3+6t^2)e^{3t+2t^3-5} = (3+6t^2)\cdot x = (3+y)x = 3x + xy$ ✓

$y' = 12t$. $72t^3/y = 72t^3/6t^2 = 12t = y'$ ✓

5. All the parts of this problem (this page and the next) concern the differential equation system $\mathbf{Y}' = A\mathbf{Y}$, where $A = \begin{bmatrix} -1 & -1/5 \\ 1/5 & -1 \end{bmatrix}$.

a. Find straight line solutions, if any

Eigenvalues of A:

$$\begin{vmatrix} -1-\lambda & -1/5 \\ 1/5 & -1-\lambda \end{vmatrix} = (1+\lambda)^2 + \frac{1}{25} = 0$$

$$(\lambda+1)^2 = -\frac{1}{25}$$

$$\lambda+1 = \pm i(1/5)$$

$$\lambda = -1 \pm i\left(\tfrac{1}{5}\right)$$

→ Since the eigenvalues are not real, there are no straight line solutions.

b. Find the general solution

Finding eigenvectors: with $\lambda = -1 + \frac{1}{5}i$ we set $(A - \lambda I)\mathbf{v} = 0$:

$$\begin{bmatrix} -1+1-\frac{1}{5}i & -\frac{1}{5} \\ \frac{1}{5} & -1+1-\frac{1}{5}i \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$$

Note row 1 $= -i$(row 2), so we have just one equation: $[-i \ \ -1]\begin{bmatrix} x \\ y \end{bmatrix} = 0$

The solution is $y = -ix$. So we take $\mathbf{v} = (1, -i)$. This leads to the solution

$$\mathbf{Y}_1 = e^{\lambda t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \left(e^{-t+\frac{t}{5}i}\right)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^{-t}\left(\cos(t/5) + i\sin(t/5)\right)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$= e^{-t}\begin{bmatrix} \cos(t/5) + i\sin(t/5) \\ \sin(t/5) - i\cos(t/5) \end{bmatrix} = e^{-t}\begin{bmatrix} \cos(t/5) \\ \sin(t/5) \end{bmatrix} + i\cdot e^{-t}\begin{bmatrix} \sin(t/5) \\ -\cos(t/5) \end{bmatrix}$$

We know the real & imaginary parts are independent solutions, so the general solution is

$$\mathbf{Y}(t) = k_1 e^{-t}\begin{bmatrix} \cos(t/5) \\ \sin(t/5) \end{bmatrix} + k_2 e^{-t}\begin{bmatrix} \sin(t/5) \\ -\cos(t/5) \end{bmatrix}.$$

c. Find an equation for the solution for which $\mathbf{Y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$Y(t) = e^{-t} k_1 \begin{bmatrix} \cos(t/5) \\ \sin(t/5) \end{bmatrix} + e^{-t} k_2 \begin{bmatrix} \sin(t/5) \\ -\cos(t/5) \end{bmatrix}$ so @ $t=0$ we get

$Y(0) = k_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} k_1 \\ -k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Therefore $k_1 = 1$, $k_2 = -2$

and

$$Y(t) = e^{-t}\left(\begin{bmatrix} \cos t/5 \\ \sin t/5 \end{bmatrix} - 2 \begin{bmatrix} \sin t/5 \\ -\cos t/5 \end{bmatrix} \right)$$

$$= e^{-t} \begin{bmatrix} \cos(t/5) - 2\sin(t/5) \\ \sin(t/5) + 2\cos(t/5) \end{bmatrix}$$

d. Give a qualitative description of all the solution curves.

all solution curves spiral inward toward $(0,0)$. The curves each cycle once around $(0,0)$ every 10π units of time.

@ $t=0$ $Y(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in part C so $Y'(0) = \begin{bmatrix} -1 & -1/5 \\ +1/5 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -7/5 \\ -9/5 \end{bmatrix}$

Since $Y(0) \times Y'(0) = \frac{-1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix} \times \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{-1}{5}\begin{vmatrix} 1 & 7 \\ 2 & 9 \end{vmatrix} = -\frac{1}{5}(9-14) = 1 > 0$

that trajectory goes counterclockwise, so all trajectories go counterclockwise.

e. Find all the equilibrium solutions and identify each as a sink, source, saddle, spiral sink, spiral source, or center.

$Y' = AY$ has an equilibrium if $AY = 0$

$\begin{bmatrix} -1 & -1/5 \\ 1/5 & -1 \end{bmatrix} Y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if & only if $Y = 0$ because $\det A \neq 0$.

So $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is only equilibrium solution. It is

a spiral sink because, as explained in part d, all solutions spiral toward $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as $t \to \infty$.

6. All the parts of this problem (this page and the next) concern the differential equation system $\mathbf{Y}' = A\mathbf{Y}$, where $A = \begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix}$.

a. Find straight line solutions, if any

eigenvalues: $|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 6 \\ 3 & -9-\lambda \end{vmatrix} = (\lambda+2)(\lambda+9) - 18 = 0$$

$\lambda^2 + 11\lambda + 18 - 18 = 0$

$\lambda^2 + 11\lambda = \lambda(\lambda+11) = 0$

so $\lambda = 0$ or -11

eigenvectors: $\lambda = 0$ $(A - 0I)V = 0$

$\begin{bmatrix} -2 & 6 \\ 3 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if & only if

$-2x + 6y = 0$ (2nd row multiple of 1st)

$x = 3y$. Choose $(3,1) = V$.

$A + 11I = \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix}$ row 1 = 3 · row 2 so $3x + 2y = 0$ take

$W = (-2, 3)$ for eigenvector.

straight line solutions are

$Y_1(t) = e^{0t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$Y_2(t) = e^{-11t}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

b. Find the general solution

General solution is

$$Y(t) = k_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-11t}$$

c. Find an equation for the solution for which $\mathbf{Y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$Y(0) = k_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-11 \cdot 0} = \begin{bmatrix} 3 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Find k_1 & k_2 using Cramer's rule:

$$k_1 = \frac{\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}} = \frac{3+4}{11} = \frac{7}{11} ; \quad k_2 = \frac{\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix}}{11} = \frac{5}{11}$$

So

$$Y(t) = \frac{7}{11} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \frac{5}{11} e^{-11t} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

d. Give a qualitative description of all the solution curves.

all solution curves are parallel to the line $y = -\frac{3}{2} x$

on each line, the solution exponentially decreases to the line's intersection with $y = 3x$. Algebraically

$Y(t) = k_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + e^{-11t} k_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ approaches $k_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ as $t \to \infty$

e. Find all the equilibrium solutions and identify each as a sink, source, saddle, spiral sink, spiral source, or center.

set $AY = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We already found in part a $AY = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ if & only if $Y = k_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ or if & only if $x_0 = 3y_0$. Every point P of the line $x = 3y$ is therefore an equilibrium. These points are not sinks, sources, saddles, spiral sinks or sources, nor centers. nearby points on the line $y = -\frac{3}{2}x$ have trajectories that approach P on that line. But a nearby point not on the line has a trajectory that approaches a different point on $x = 3y$, so does not approach P.

7. The matrix $A = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix}$ has eigenvalues -1 and 3, with corresponding eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Find e^{At}.

$$e^{At} = e^{PDP^{-1}t} = Pe^{Dt}P^{-1}$$

where $P = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$. So

$$e^{At} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}^{-1}$$

Now $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ so

$$P^{-1} = \frac{1}{1-2} \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{+3t} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -e^{-t} & 2e^{-t} \\ e^{3t} & -e^{3t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{3t} - e^{-t} & 2e^{-t} - 2e^{3t} \\ -e^{-t} + e^{3t} & 2e^{-t} - e^{3t} \end{bmatrix}$$

8. The matrix $A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$ has the repeated eigenvalue 3, and all eigenvectors are multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find e^{At}.

$A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Must find vector $w = \begin{bmatrix} a \\ b \end{bmatrix}$ such that

$A\begin{bmatrix} 1 & a \\ 2 & b \end{bmatrix} = \begin{bmatrix} 1 & a \\ 2 & b \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. This shows $A\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 3\begin{bmatrix} a \\ b \end{bmatrix}$

so we solve $(A-3I)\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 5-3 & -1 \\ 4 & 1-3 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

By inspection $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a solution. Now we have

$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$ and $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{1-2}\begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$

Therefore $e^{At} = e^{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}t\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} e^{\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}t} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$

$= \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} e^{3t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2t-1 & t+1 \\ 2 & 1 \end{bmatrix}$

$= e^{3t}\begin{bmatrix} 2t+1 & t+2 \\ 4t & 2t+3 \end{bmatrix}$.

9. Theorems etc

a. State the linearity principle for systems of the form $\mathbf{Y}' = A\mathbf{Y}$ with A a constant 2×2 matrix.

See page 249

b. Complete the statement of the theorem that begins *Suppose the matrix A has real eigenvalue λ with associated eigenvector* $\mathbf{V}$.

See page 274

c. State the theorem giving the general solution of a two dimensional linear system with a repeated eigenvalue.

See page 319

d. Tell how initial value problems for a system $\mathbf{Y}' = A\mathbf{Y}$ can be solved using the exponential matrix e^{At}.

If $Y(0) = Y_0$ the solution to the I.V.P. is $e^{At} \cdot Y_0$.

e. What is the definition of e^B for an $n \times n$ matrix B?

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \cdots$$

$$= I + B + \frac{1}{2!}B^2 + \frac{1}{3!}B^3 + \frac{1}{4!}B^4 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!} B^k$$