# The Annihilator Method:
# An Alternative to Undetermined Coefficients

## Introduction

In section 4.1 of our text, a method is presented for solving a differential equation of the form

$$y''+py'+qy=g(t). \tag{1}$$

where $p$ and $q$ are constants and $g$ is some function of $t$. The method only works when $g$ is of a particular form, and by guessing a linear combination of such forms, it is possible to determine the coefficients that make the proposed solution satisfy the equation. In these notes I will present an alternative approach called the Annihilator Method.

**Differential Operators.** As a first step, we consider the differential operator $D$ defined by the equation $Df(t) = f'(t)$. With this notation, $y' = Dy$ and $y'' = D^2y$. We can combine these operations. For example, $(D^2 + 3D + 6)$ represents an operator that transforms a function $y$ to $(D^2 + 3D + 6)y = D^2y + 3Dy + 6y = y'' + 3y' + 6y$. These notational conventions allows us to rewrite equation (1)

$$(D^2+pD+q)y=g(t). \tag{2}$$

The expression in the parentheses is a polynomial in $D$, called the characteristic polynomial of the differential equation. As we have already seen, the roots of the characteristic polynomial give us solutions to the homogeneous equation. Let's look at this in a new way, using operators.

Example: $y''-3y'-4y=0$. In operator notation the equation is $(D^2-3D-4)y=0$. Now factor the polynomial in $D$, to find $(D-4)(D+1)y=0$. This shows that there are eigenvalues of -1 and 4, so $e^{-t}$ and $e^{4t}$ are solutions to the differential equation. The new way to look at this is to associate each solution with one factor of the polynomial. The solution $e^{-t}$ goes with $(D + 1)$ and $(D + 1)\, e^{-t} = 0$. If we apply $(D - 4)$ to both sides of the equation we derive $(D - 4)\,(D + 1)\, e^{-t} = (D - 4)0 = 0$. In this way we see that a solution to the homogeneous equation for a single factor of the characteristic polynomial is also a solution to the homogeneous equation for the full polynomial. Also, the factors of the polynomial are interchangeable: $(D - 4)\,(D + 1) = (D + 1)\,(D - 4)$, so we can repeat the preceding argument with $(D - 4)$ and show that $e^{4t}$ is also a solution to the homogeneous equation for the full polynomial.

With this new way of looking at things, we see that solutions to a homogeneous linear differential equation can be found by factoring the characteristic polynomial in $D$ and finding homogeneous solutions for each factor. In general, the factor $(D - r)$ always gives a solution $e^{rt}$ because $(D - r)\, e^{rt} = 0$. Here we adopt another new (and suggestive) terminology. We say the operator $(D - r)$ *annihilates* $e^{rt}$ because the operator transforms $e^{rt}$ to 0 (nothing).

**Annihilating $g(t)$.** The annihilator method transforms an inhomogeneous equation to a homogeneous equation by applying an operator on both sides. Here is an example: Find the general solution of $(D^2-3D-4)y=5e^{-2t}$. Notice that $(D + 2)$ annihilates $5e^{-2t}$ because $(D + 2)\, 5e^{-2t} = 0$. Applying $(D+2)$ to both sides of $(D^2-3D-4)y=5e^{-2t}$ gives us $(D+2)(D^2-3D-4)y=(D+2)5e^{-2t}=0$. Factoring the quadratic part we get $(D+2)(D-4)(D+1)y=0$. Now we have a homogeneous equation and know how to get the general solution: $y = c_1e^{-2t} + c_2e^{4t} + c_3e^{-t}$.

Note that while every particular solution of the original equation must be of the form just derived, the reverse is not true – not everything of that form is a particular solution of the original equation. For example, if we take $c_1 = c_2 = 0$ and $c_3 = 1$, we get $y = e^{-t}$, and since that is a solution to the homogeneous equation $(D^2 - 3D - 4)y = 0$, it cannot be a solution to $(D^2 - 3D - 4)y = 5e^{-2t}$. But we now know that if there is a particular solution, it has to be expressible as $y = c_1e^{-2t} + c_2e^{4t} + c_3e^{-t}$ with some specific choice of the coefficients. Moreover, we might as well take $c_2 = c_3 = 0$ because $c_2e^{4t} + c_3e^{-t}$ is the general solution of the homogeneous equation, and so is annihilated by the differential operator on the left side of the equation. That means it cannot contribute anything toward our goal of producing $5e^{-2t}$ on the right side. All that remains is $c_1$. In this way, we deduce that a particular solution MUST be of the form $c_1e^{-2t}$. We still have to find $c_1$, but we have determined that this is the only possibility, and have not had to guess.

**Annihilators for various functions.** What functions can be annihilated? Just those that arise as solutions to homogeneous constant coefficient linear differential equations. We have seen that functions of the form $ce^{rt}$ can be annihilated by $D - r$. Working out the details with complex eigenvalues we find that $be^{\alpha t}(\cos \beta t)$ and $ce^{\alpha t}(\sin \beta t)$ are both annihilated by $(D^2 - 2\alpha D + \alpha^2 + \beta^2) = (D - \alpha + \beta i)(D - \alpha - \beta i)$. We have also seen that when $r$ is a repeated eigenvalue, we get solutions of the form $ce^{rt}$ and $cte^{rt}$. Therefore, $cte^{rt}$ is annihilated by $(D - r)^2$. More generally, for any polynomial $f(t)$ of degree $m$, the function $f(t)e^{rt}$ is annihilated by $(D - r)^{m+1}$. This can be extended to the complex case, or specialized to the case $r = 0$. On one hand, we find that the functions $f(t)e^{\alpha t}(\cos \beta t)$ and $f(t)e^{\alpha t}(\sin \beta t)$ are annihilated by $(D^2 - 2\alpha D + \alpha^2 + \beta^2)^{m+1}$, and on the other hand we find that $f(t)$ itself is annihilated by $D^{m+1}$. These results are summarized in the table below.

**Table of Annihilators**

| $f(t)$ | **Annihilator** |
|---|---|
| $a_m t^m + a_{m-1}t^{m-1} + \cdots + a_1 t + a_0$ | $D^{m+1}$ |
| $e^{rt}$ | $D - r$ |
| $t^m e^{rt}$ | $(D - r)^{m+1}$ |
| $\cos \beta t$ or $\sin \beta t$ | $D^2 + \beta^2$ |
| $t^m \cos \beta t$ or $t^m \sin \beta t$ | $(D^2 + \beta^2)^{m+1}$ |
| $e^{\alpha t} \cos \beta t$ or $e^{\alpha t} \sin \beta t$ | $(D - \alpha)^2 + \beta^2 = D^2 - 2\alpha D + \alpha^2 + \beta^2$ |
| $t^m e^{\alpha t} \cos \beta t$ or $t^m e^{\beta t} \sin \beta t$ | $\left[(D - \alpha)^2 + \beta^2\right]^{m+1} = \left[D^2 - 2\alpha D + \alpha^2 + \beta^2\right]^{m+1}$ |

For any multiple of a function in the first column the appropriate annihilator is shown in the second column. You can also use the annihilators for linear combinations. For example, to annihilate $g(t) = 3t + 5e^{-2t}$, you *multiply* the annihilators of the individual terms: $D^2 (D + 2)$. This works when the individual annihilators have distinct eigenvalues. But suppose you want to annihilate $g(t) = te^{4t} + 6t^3e^{4t}$. For the first term the annihilator is $(D - 4)^2$ while for the second it is $(D - 4)^4$. We don't have to multiply these, because one is a divisor of the other. Instead we take the larger exponent, in this case 4. That is, since $(D-4)^4$ annihilates both $te^{4t}$ and $6t^3e^{4t}$, it annihilates their sum.

**Annihilator Method Algorithm**

The table above gives you a complete catalog of functions that can be annihilated, and it tells you what the annihilator is of each. How do we use that to find particular solutions to a differential equation? Here are step by step instructions.

1. Express the differential equation in operator form $h(D)y = g(t)$, where $h(D)$ is a polynomial.
2. Factor $h(D)$ into linear factors for real eigenvalues and quadratic factors for complex (nonreal) eigenvalues.
3. Form the general solution to the homogeneous equation $h(D)y = 0$.
4. Find an annihilator for $g(t)$. We'll call that $k(D)$. It is a polynomial and will already be in factored form.
5. Form the general solution to the homogeneous equation $k(D)h(D)y = 0$.
6. Remove from this second general solution the functions that already appeared in the general solution of the original homogeneous equation.
7. Substitute the remaining general solution into the original equation $h(D)y = g(t)$ and solve for the unknown coefficients.

In the end, you still have to solve for the coefficients of the specified functions, just as in the method of undetermined coefficients. In fact, the two methods work for exactly the same types of equations, and use the same steps to find the coefficients. But the annihilator method is better for two reasons. First, you know ahead of time exactly what type of function $g(t)$ must be for the method to succeed: any linear combination of functions given in the first column of the table. Second, you don't have to guess the correct form for the particular solution. That is generated automatically when you use the annihilator method.

**More Examples.**

1. $(D^2-3D-4)y = 10e^{4t}$. We factor the polynomial in $D$ first, finding $(D-4)(D+1)y = 10e^{4t}$, and recognize that the general solution to the homogeneous equation is $y = c_1e^{4t} + c_2e^{-t}$. Next, we recognize that the annihilator of $g(t) = 10e^{4t}$ is $(D-4)$. Applying this to both sides of the original differential equation we get $(D-4)^2(D+1)y = 0$. The general solution to this homogeneous equation is $y = c_1e^{4t} + c_3\,te^{4t} + c_2\,e^{-t}$. We eliminate the terms we had before, leaving $y = c_3\,te^{4t}$. This is the form of the particular solution we need. Substituting into the original differential equation, we have

$$
\begin{aligned}
(D^2-3D-4)\,c_3\,te^{4t} &= 10e^{4t}\\
c_3((te^{4t})'' - 3(te^{4t})' - 4(te^{4t})) &= 10e^{4t}\\
c_3((16te^{4t}+8e^{4t}) - 3(4te^{4t}+e^{4t}) - 4(te^{4t})) &= 10e^{4t}\\
c_3((16te^{4t} - 12te^{4t} - 4te^{4t}) + (8e^{4t} - 3e^{4t})) &= 10e^{4t}\\
c_3(5e^{4t}) &= 10e^{4t}\\
c_3 &= 2
\end{aligned}
$$

This gives us the particular solution $y_{\text{part}}(t) = 2te^{4t}$, and the general solution to the original equation is $y_{\text{hom}}(t) + y_{\text{part}}(t) = c_1e^{4t} + c_2e^{-t} + 2te^{4t}$.

2. $(D^2 - 3D - 4)y = 3t^2$. We begin as before, by finding the general solution to the homogeneous equation, $y_{\text{hom}}(t) = c_1e^{4t} + c_2e^{-t}$. Next, we recognize that the annihilator of $g(t) = 3t^2$ is $D^3$. Applying this to both sides of the original differential equation we get $D^3(D-4)(D+1)y = 0$. The general solution to this homogeneous equation is $y = c_1e^{4t} + c_2 e^{-t} + c_3 + c_4t + c_5t^2$. We again eliminate the terms in the original homogeneous equation solution, leaving $y = c_3 + c_4t + c_5t^2$. This is the form of the particular solution we need. Substituting into the original differential equation, we have

$$\begin{aligned}
(D^2 - 3D - 4)(c_3 + c_4t + c_5t^2) &= 3t^2 \\
(c_3 + c_4t + c_5t^2)'' - 3(c_3 + c_4t + c_5t^2)' - 4(c_3 + c_4t + c_5t^2) &= 3t^2 \\
(2c_5) - 3(c_4 + 2c_5t) - 4(c_3 + c_4t + c_5t^2) &= 3t^2 \\
(2c_5 - 3c_4 - 4c_3) + t(-6c_5 - 4c_4) + t^2(-4c_5) &= 3t^2
\end{aligned}$$

Equating coefficients of powers of $t$ on each side gives us three equations:

$$\begin{aligned}
2c_5 - 3c_4 - 4c_3 &= 0 \\
-6c_5 - 4c_4 &= 0 \\
-4c_5 &= 3
\end{aligned}$$

and since this is a triangular linear system, it is not difficult to find the solution: $c_5 = -3/4$, $c_4 = 9/8$, $c_3 = -39/32$. This gives us the particular solution $y_{\text{part}}(t) = -39/32 + (9/8)t + (-3/4)t^2$, and the general solution to the original equation is $y_{\text{hom}}(t) + y_{\text{part}}(t) = c_1e^{4t} + c_2e^{-t} - 39/32 + (9/8)t + (-3/4)t^2$.

Exercises: For all the assigned problems in section 4.1, use the annihilator method to find the form of the particular solution as a linear combination of specific functions.