

Name \_\_\_\_\_

**Instructions:** Read the questions carefully, and be sure to answer all parts of each question. For full credit on non-essay questions, you must show work or give some explanation to verify your answer. Where possible, check that your solution is correct. You must communicate to me how you reached your answer and why you believe it is right.

1. Use numerical and/or graphical methods to investigate whether or not the following limit exists:

$$\lim_{x \rightarrow 4} \frac{x^2 - 4x}{x^2 - 5x + 4}$$

(That would be entered as  $(x^2 - 4x)/(x^2 - 5x + 4)$  on a calculator.) If you feel the limit exists, give its approximate value. If you feel the limit does not exist, tell why. In either case, show me the evidence that justifies your conclusion, and describe how you found that evidence.

Here is a table of values for  $x$  near 4

$x$	3.9	3.99	3.999	4.001	4.01	4.1
$\frac{x^2 - 4x}{x^2 - 5x + 4}$	1.3448	1.3344	1.3334	1.3332	1.3322	1.3226

From these values it appears  $\frac{x^2 - 4x}{x^2 - 5x + 4}$  should

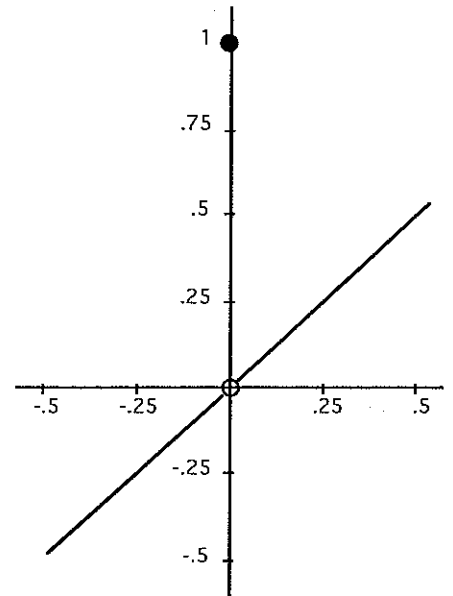
be close to 1.333... when  $x$  is close to 4. In fact

for  $3.999 < x < 4.001$  it appears  $1.3332 < \frac{x^2 - 4x}{x^2 - 5x + 4} < 1.3334$ .

Conclusion: The limit does appear to exist

and  $= 1.333...$

For the next four problems,  $f(x) = x + \text{int}(1 - x^2)$ . (Recall that  $\text{int}$  of any positive number is found by dropping all digits to the right of the decimal point, so for example,  $\text{int}(2.36) = 2$ ). The figure at right shows the graph of  $f(x)$  for points near  $x = 0$ . Based on this graph, answer the following questions. Be sure to justify each answer.



2. Does  $f(0)$  exist? If so, give its numerical value.

$$\begin{aligned} f(0) &= 0 + \text{int}(1 - 0^2) \\ &= 0 + \text{int}(1) = 1 \end{aligned}$$

it exists and  $= 1$ .

3. Does  $\lim_{x \rightarrow 0} f(x)$  exist? If so, give its numerical value.

Based on the graph  $\lim_{x \rightarrow 0} f(x)$  does appear to exist and equals 0. As  $x$  values very close to (but not = to) 0 are chosen, the  $f(x)$  values fall on the line  $f(x) = x$  and so they are also close to 0.

4. Does  $\lim_{x \rightarrow 0^+} f(x)$  exist? If so, give its numerical value.

The limit from the left exists and  $= 0$  because the limit from both sides  $= 0$ .

5. Is  $f(x)$  continuous at  $x = 0$ ?

$$\text{Since } f(0) = 1 \neq 0 = \lim_{x \rightarrow 0} f(x)$$

This limit cannot be found by direct substitution. This shows  $f(x)$  is not continuous @  $x = 0$ .

For the two problems on this page, find an exact value for the given limit, using theoretical or algebraic methods. Also give some justification for your method.

$$6. \lim_{x \rightarrow 3} \frac{\sqrt{4x-3}}{4-x}$$

$4x-3$  is a polynomial so  $\lim_{x \rightarrow 3} 4x-3 = 9$   $= 4 \cdot 3 - 3$

$4-x$  " " " "  $\lim_{x \rightarrow 3} 4-x = 4-3 = 1$

So, since  $\lim_{x \rightarrow 3} 4x-3$  exists and is positive,

$$\text{we see } \lim_{x \rightarrow 3} \sqrt{4x-3} = \sqrt{\lim_{x \rightarrow 3} 4x-3} = \sqrt{9} = 3$$

Then since  $\lim_{x \rightarrow 3} \sqrt{4x-3}$  exists and  $= 3$

and since  $\lim_{x \rightarrow 3} 4-x$  exists and  $= 1 \neq 0$

$$\text{we conclude } \lim_{x \rightarrow 3} \frac{\sqrt{4x-3}}{4-x} = \frac{3}{1} = 3$$

$$7. \lim_{x \rightarrow 4} \frac{\sqrt{4x}-4}{x-4}$$

as  $x \rightarrow 4$  both  $\sqrt{4x}-4$  and  $x-4 \rightarrow 0$   
so direct substitution is invalid.

$$\begin{aligned} \text{Algebraically, } \frac{\sqrt{4x}-4}{x-4} &= \frac{\sqrt{4x}-4}{x-4} \cdot \frac{\sqrt{4x}+4}{\sqrt{4x}+4} = \frac{4x-16}{(x-4)(\sqrt{4x}+4)} \\ &= \frac{4(x-4)}{(x-4)(\sqrt{4x}+4)} = \frac{4}{\sqrt{4x}+4}. \end{aligned}$$

This is true for all  $x \neq 4$

So

$$\lim_{x \rightarrow 4} \frac{\sqrt{4x}-4}{x-4} = \lim_{x \rightarrow 4} \frac{4}{\sqrt{4x}+4} = \frac{4}{\sqrt{4 \cdot 4}+4} = \frac{4}{4+4} = \frac{1}{2}$$

This is valid because direct substitution is valid  
by similar steps as used in ~~problem~~ #6.

## 8. Derivative of a function.

a. State the definition of derivative of a function  $f$  at a number  $a$ .

The derivative of  $f$  at  $a$  is

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

provided this limit exists.

Also correct:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

b. Use your definition to find the derivative of  $f(x) = 6/x$  at  $x = 2$ . Show all your work and explain briefly what you are doing.

We want to find  $\lim_{x \rightarrow 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \rightarrow 2} \frac{6/x - 6/2}{x - 2}$

Before computing this limit, use algebra to simplify  $\frac{6/x - 6/2}{x - 2}$

$$\frac{\frac{6}{x} - \frac{6}{2}}{x - 2} = \frac{\frac{6}{x} - 3}{x - 2} = \frac{\frac{6}{x} - 3}{x - 2} \cdot \frac{x}{x} = \frac{6 - 3x}{x(x - 2)} = \frac{3(2 - x)}{x(x - 2)}$$

Next, notice  $-(2 - x) = -2 - (-x) = -2 + x = x - 2$

so  $(2 - x) = -(x - 2)$ . Substitute

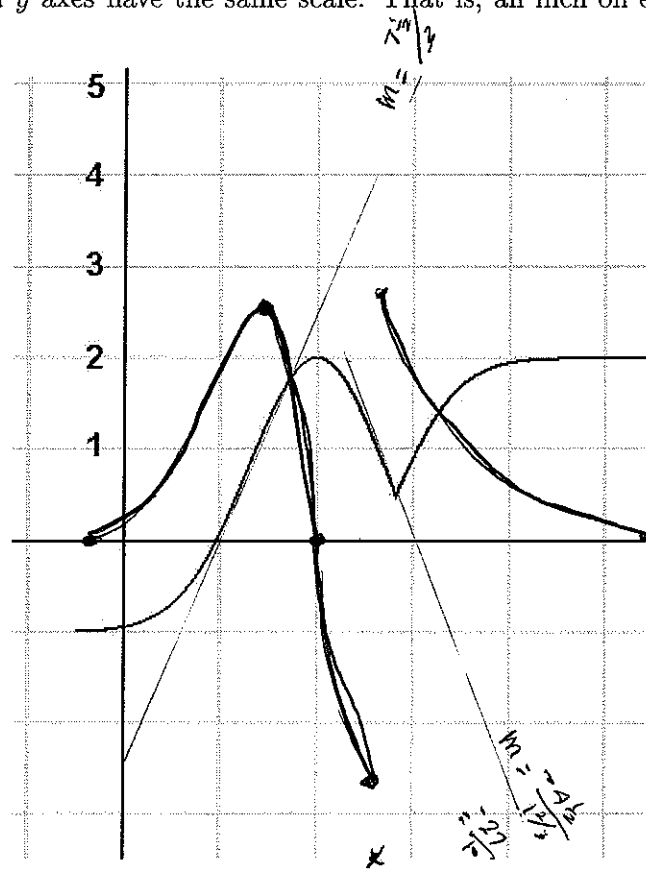
$$\frac{\frac{6}{x} - \frac{6}{2}}{x - 2} = \frac{3(2 - x)}{x(x - 2)} = \frac{3 \cdot -(x - 2)}{x(x - 2)} = \frac{-3}{x}$$

Now  $\lim_{x \rightarrow 2} \frac{6/x - 6/2}{x - 2} = \lim_{x \rightarrow 2} \frac{-3}{x} = \boxed{\frac{-3}{2}}$

by direct substitution. That's valid because

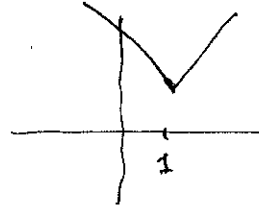
$\frac{-3}{x}$  is a rational function and  $x = 2$  is in the domain (all  $x$  except  $x = 0$ )

9. The figure below shows the graph of the function  $f$ . Add to the figure a sketch of the graph of the derivative  $f'$ . Assume that the  $x$  and  $y$  axes have the same scale. That is, an inch on each axis represents the same number of units.



10. Suppose a function  $f(x)$  is continuous at  $x = 1$ . Must the derivative of  $f(x)$  also exist at  $x = 1$ ? Suppose a function  $g(x)$  has a derivative at  $x = 1$ . Must  $g(x)$  also be continuous at  $x = 1$ ? For each question, either explain why the answer is yes, or give an example of a particular function to show that the answer is no. Your example can be either in the form of a graph or an equation.

Q1: NO. Example -- this function is continuous at  $x=1$  but has no derivative there:



Q2: yes. We have a theorem that says

if  $f'(1)$  exists,  $f$  is continuous at 1.

11. Is the function  $f(x) = \frac{1}{x^2 - 4}$  continuous in the closed interval  $[-1, 2]$ ? Justify your answer.

$\frac{1}{x^2 - 4}$  is a rational function, defined except where  $x^2 - 4 = 0$ . So for  $x = 2$  or  $-2$   $f(x)$  is undefined. At all other  $x$ ,  $f(x)$  is continuous. Since  $f(x)$  is not continuous at every point of  $[-1, 2]$  (specifically, not at ~~left~~ <sup>right</sup> endpoint)  $f(x)$  is not continuous in that interval.

12. For the most part, the functions in calculus which are defined by some sort of algebraic equation are either continuous for all real numbers, or fail to be continuous at some special points in a predictable fashion. Give a brief description of two things which you might see in an equation that would warn you about a possible point of discontinuity. For each one, also give an example, and explain where your example is continuous and where it fails to be continuous.

1. A denominator that might  $= 0$  for some  $x$ 's  
 $f(x) = \frac{1}{x}$  continuous every where except  $x = 0$
2. A  $\sqrt{\quad}$  (or  $\sqrt[3]{\quad}$ ,  $\sqrt[n]{\quad}$  etc) that might have a negative value inside for some  $x$ 's  
 $f(x) = \sqrt{x}$  continuous for  $0 \leq x$
3. A  $\ln(\quad)$  or  $\log(\quad)$  that might have 0 or negative value in  $(\quad)$ 's for some  $x$ 's.  
 $f(x) = \ln(x)$  continuous for  $x > 0$
4.  $\sin^{-1}$  or  $\cos^{-1}$  of something that might be outside the range  $[-1, 1]$  for some  $x$ 's  
 $\sin^{-1}(x)$  continuous for  $-1 \leq x \leq 1$

Finding Derivatives. For

each part find the derivative of the given function. You may use any rules or techniques we have studied. For each step either show your work or state what you did.

a.  $f(x) = 4x^2 - 4/x^2$

$$f(x) = 4x^2 - 4x^{-2}$$

$$f'(x) = 8x - 4(-2)x^{-3}$$

$$= 8x + 8x^{-3}$$

$$= 8x + \frac{8}{x^3}$$

b.  $f(x) = \frac{e^x + 4}{e^x - 4}$

Quotient rule:

$$f'(x) = \frac{(e^x + 4)'(e^x - 4) - (e^x - 4)'(e^x + 4)}{(e^x - 4)^2}$$

$$= \frac{e^x(e^x - 4) - e^x(e^x + 4)}{(e^x - 4)^2}$$

$$= \frac{e^{2x} - 4e^x - e^{2x} - 4e^x}{(e^x - 4)^2} = \frac{-8e^x}{(e^x - 4)^2}$$