

Linearization and Differential Equations

Plotting an Approximate Solution Curve

As we saw in the unit on exponential growth, population models sometimes assume a constant relative growth rate. That leads to an equation of the form $P = P_0 e^{kt}$ for the size of the population at time t . But this is unrealistic in the long run, because an exponential equation exhibits unlimited growth, and that is not possible for real populations. In light of this, we can modify our model. Instead of assuming the relative growth rate is a constant, assume that it changes with the size of the population, so that larger populations will grow with a smaller relative rate. That leads to a differential equation of the form $P' / P = k(P)$ where k is some decreasing function of P . A simple version of this idea makes $k(P)$ a linear function. In this handout you will see how linearization can be used to approximate a solution to one particular differential equation based on these ideas. For simplicity of notation, the population variable will be represented by y (in units of 1000's) and the time variable will be x (in units of years). The differential equation we will consider is

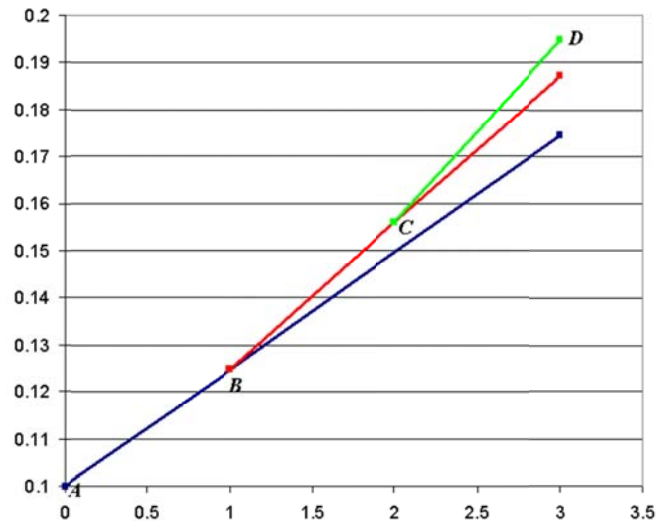
$$y' = .01(25 - y)y. \quad (1)$$

We also assume an initial population of $y = .1$ at time $x = 0$.

Without solving this equation, we can plot an approximate solution curve as follows. We begin at the point $(x,y) = (0,.1)$, labeled A on the graph below. At that point the slope of the solution curve is given by $y' = .01(25 - y)y = .01(25 - .1)(.1) = .0249$. This allows us to find the linearization or tangent line at point A . It has equation $L(x) = .1 + .0249(x - 0)$, and is represented by the straight line through A and B in the graph. Using $x = 1$ in the equation gives us $L(1) = .1 + .0249 = .1249$.

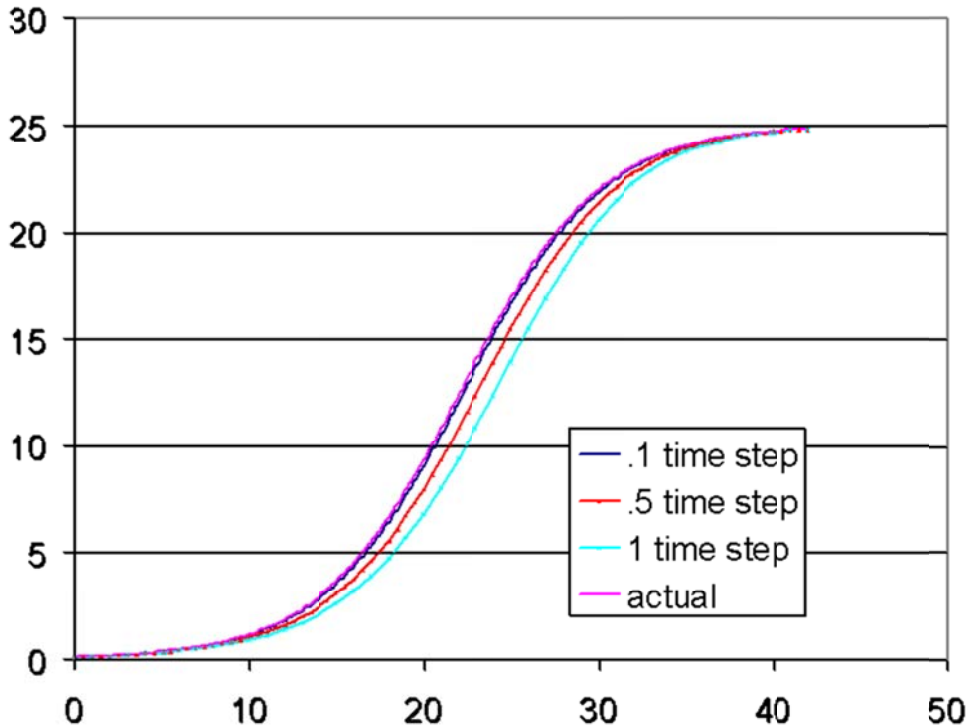
This is an approximation to the true value of y for $x = 1$. So we plot the point $B = (1,.1249)$. Although we know this is not exactly on the solution curve, it should be fairly close to the curve. This is the same logic that is used when we linearize a function at a point. The difference is, this time we do not have an equation for the function.

Now that we have point B , we can repeat the process to find another approximate point on the curve. We substitute the y value at point B into equation (1) and compute a new value of y' , (it comes out to approximately .03107 this time), and that in turn gives us a new linearization $L(x)$. This time we are finding a tangent line at point B and it is a little steeper than the first line. It appears in the figure as the line through B and C . Taking $x = 2$ in this new $L(x)$ gives us the point C . Repeating again, we find a linearization at C , and that gives us a point at D . We can continue for as many points as we wish, generating at least an estimate for points on the true solution curve.



While this would be tedious to do by hand, it is very easy using appropriate software. Using Excel, for example, we can automatically generate each new x and y using prior values. Excel will also graph the resulting pairs (x,y) for us. See the next page for some graphical results. Note that we can generate our data points with closer spacing than described above. In the earlier discussion we generated points for $x = 1, 2, 3, \dots$. We say that these points generated using a spacing of $\Delta x = 1$. We would expect to get greater accuracy using a smaller step size. That proves to be the case in the graph below.

The graph below compares approximation curves using three different choices for Δx , 1, .5, and .1, extending each curve to $x = 42$. In addition, the actual solution curve to the differential equation is shown. It is almost indistinguishable from the curve with $\Delta x = .1$.



For this example, it is possible to find an exact equation for the solution curve. It is given by

$$y = \frac{25}{1 + 249e^{-25x}}.$$

That allows us to see how accurate the approximate solution curves are. But in many cases it is impossible to find an exact equation for the solution curve. In that case, the approximate curves that can be found using linearization are very useful. This application of linearization is referred to as Euler's Method. It is described in more detail in section 9.2 of our text.

Here is a part of the Excel Table that was used to create the graphs above:

x value	approx y value	y' value	x value	approx y value	y' value	x value	approx y value	y' value
0	0.1	0.0249	7	0.47114	0.11557	14	2.12473	0.48604
1	0.1249	0.03107	8	0.58671	0.14324	15	2.61077	0.58453
2	0.15597	0.03875	9	0.72995	0.17716	16	3.1953	0.69673
3	0.19472	0.0483	10	0.9071	0.21855	17	3.89202	0.82153
4	0.24302	0.06016	11	1.12565	0.26874	18	4.71355	0.95621
5	0.30318	0.07488	12	1.39439	0.32916	19	5.66976	1.09598
6	0.37806	0.09309	13	1.72355	0.40118	20	6.76574	1.23368