Linearization and Differential Equations
Plotting an Approximate Solution Curve

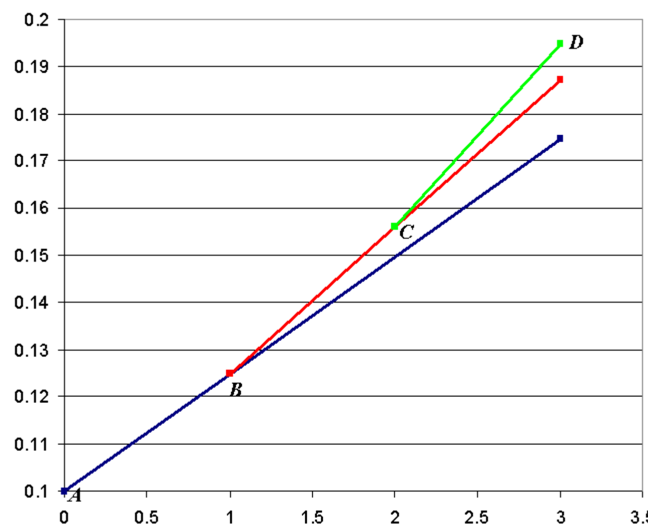
In the development of population models, it is natural initially to assume a constant relative growth rate. That means assuming $P'(t) / P(t)$ is constant, where $P(t)$ represents the population at time t. For such a model, doubling the population size would also double the rate of growth. That is what would be expected if you assume that at any given time, a fixed percentage of the population will reproduce.

Assuming a constant $P'(t) / P(t)$ leads to a differential equation with solutions of the form $P = P_0 e^{kt}$. But this is unrealistic in the long run, because an exponential equation exhibits unlimited growth, and that is not possible for real populations. In light of this, we can modify our model. Instead of assuming the relative growth rate is a constant, assume that it changes with the size of the population, so that larger populations will grow with a smaller relative rate. That leads to a differential equation of the form $P' / P = k(P)$ where k is some decreasing function of P. A simple version of this idea makes $k(P)$ a linear function. In this handout you will see how linearization can be used to approximate a solution to one particular differential equation based on these ideas. For simplicity of notation, the population variable will be represented by y (in units of 1000's) and the time variable will be x (in units of years). The differential equation we will consider is

$$y' = .01(25 - y)y. \qquad (1)$$

We also assume an initial population of $y = .1$ at time $x = 0$.

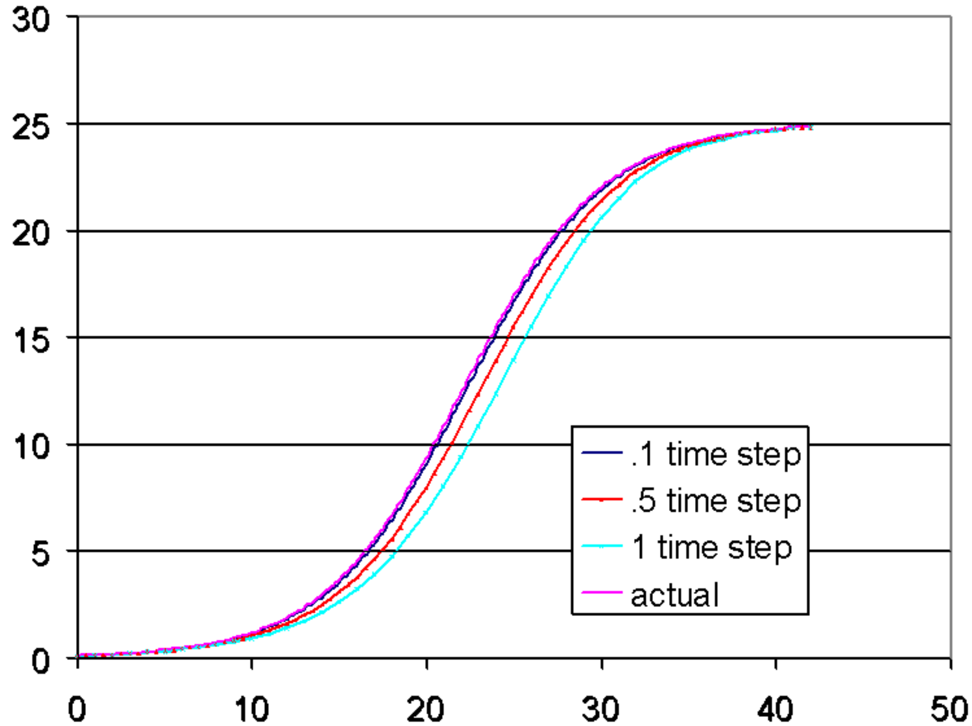
Without solving this equation, we can plot an approximate solution curve as follows. We begin at the point $(x,y) = (0,.1)$, labeled A on the graph below. At that point the slope of the solution curve is given by $y' = .01(25 - y)y = .01(25-.1)(.1) = .0249$. This allows us to find the linearization or tangent line at point A. It has equation $L(x) = .1 + .0249(x - 0)$, and is represented by the straight line through A and B in the graph. Using $x = 1$ in the equation gives us $L(1) = .1 + .0249 = .1249$. This is an approximation to the true value of y for $x = 1$. So we plot the point $B = (1,.1249)$. Although we know this is not exactly on the solution curve, it should be fairly close to the curve. This is the same logic that is used when we linearize a function at a point. The difference is, this time we do not have an equation for the function.



Now that we have point B, we can repeat the process to find another approximate point on the curve. We substitute the y value at point B into equation (1) and compute a new value of y', (it comes out to approximately .03107 this time), and that in turn gives us a new linearization $L(x)$. This time we are finding a tangent line at point B and it is a little steeper than the first line. It appears in the figure as the line through B and C. Taking $x = 2$ in this new $L(x)$ gives us the point C. Repeating again, we find a linearization at C, and that gives us a point at D. We can continue for as many points as we wish, generating at least an estimate for points on the true solution curve.

While this would be tedious to do by hand, it is very easy using appropriate software. Using Excel, for example, we can automatically generate each new x and y using prior values. Excel will also graph the resulting pairs (x,y) for us. See the next page for some graphical results. Note that we can generate our data points with closer spacing than described above. Before, we generated points for $x = 1, 2, 3, \ldots$. We say that these points were generated using a step size of $\Delta x = 1$. We would expect to get greater accuracy using a smaller step size. That proves to be the case in the graph below.

The graph below compares approximation curves using three different choices for Δx, 1, .5, and .1, extending each curve to $x = 42$. In addition, the actual solution curve to the differential equation is shown. It is almost indistinguishable from the curve with $\Delta x = .1$.



For this example, it is possible to find an exact equation for the solution curve. It is given by

$$y = \frac{25}{1 + 249\, e^{-.25x}}.$$

That allows us to see how accurate the approximate solution curves are. But in many cases it is impossible to find an exact equation for the solution curve. In that case, the approximate curves that can be found using linearization are very useful. This application of linearization is referred to as Euler's Method. It is described in more detail in section 11.3 of our text.

Here is a part of the Excel Table that was used to create the graphs above:

x value	approx y value	y' value
0	0.1	0.0249
1	0.1249	0.03107
2	0.15597	0.03875
3	0.19472	0.0483
4	0.24302	0.06016
5	0.30318	0.07488
6	0.37806	0.09309

x value	approx y value	y' value
7	0.47114	0.11557
8	0.58671	0.14324
9	0.72995	0.17716
10	0.9071	0.21855
11	1.12565	0.26874
12	1.39439	0.32916
13	1.72355	0.40118

x value	approx y value	y' value
14	2.12473	0.48604
15	2.61077	0.58453
16	3.1953	0.69673
17	3.89202	0.82153
18	4.71355	0.95621
19	5.66976	1.09598
20	6.76574	1.23368

Exercises

Use the excel spreadsheet located at

http://www.dankalman.net/AUhome/classes/classesF17/calc1/handouts/eulers_method.xls

to find answers for the following problems.

1. For the differential equation described above, and with an initial population of $y = 5$, compare the approximate solution curves produced by Euler's Method with a step size of 1, 0.5, and 0.1. For each step size estimate the value of y corresponding to $x = 10$.

2. Using three different step sizes, estimate the value of y corresponding to $x = 20$. How accurate do you think your best estimate is? Why?

3. With an initial $y = 5$, the exact solution to the differential equation is given by $y = \frac{25}{1+4e^{-.25x}}$. Use this to compute y when $x = 20$, and then report on the accuracy of your best estimate from problem 2.