Exponential Growth Functions of the Form $A(t) = A_0 b^{t/d}$

In section 3.8 of our text, exponential growth models are expressed in terms of the base *e* of natural logarithms. We see equations of the form $A(t) = A_0 e^{kt}$ where A_0 and k are fixed constants. In fact, each constant represents a specific aspect of the function. The first constant, A_0, is the initial value $A(0)$, which also can be interpreted as the *y* intercept on the graph of the function. The second, k, is the relative instantaneous growth or decay rate $A'(t) \div A(t)$, which is constant for an exponential function. Models are sometimes specified in terms of these meanings. For example, if we are describing a population of size $P(t)$ at time t, we might find that the initial population size is 5000 and relative growth rate is 1.06. That means we can immediately write the equation as $P(t) = 5000e^{1.06t}$.

There is a second useful form for an exponential growth function, $A(t) = A_0 b^{t/d}$. Here there are three specific constants, A_0, b and d, with interpretations specified in the following statement:

The function $A(t) = A_0 b^{t/d}$ equals A_0 at time 0 and is multiplied by b every d units of time.

For example, consider a mutant space blob that is 1000 kg at time 0, and which doubles in size every 6.5 hours. If $A(t)$ represents the mass (in kg) of the blob at time t (in units of hours), then we have the equation $A(t) = 1000\ 2^{t/6.5}$, because doubling means the same thing as being multiplied by 2. Or, as another example, suppose you invest $60,000 in a mutual fund that is expected to appreciate by 1.5% every three months. Observe that if a starting value of V appreciates by 1.5%, then it grows to $V + .015V = 1.015V$. This shows that 1.5% appreciation amounts to *multiplication* by 1.015. Therefore, we can write the equation

$$V(t) = 60{,}000 \cdot\ 1.015^{t/3}$$

For the value of the fund, in dollars, after t months. As one more example, consider the radioactive element radium-226 which has a half life of 1590 years. That means that the amount of radium-226 in any specific object decreases by half (or in other words, is multiplied by 1/2) every 1590 years. Therefore, the amount of radium in the sample after t years is given by $A(t) = A_0 \left(\frac{1}{2}\right)^{t/6.5}$ where A_0 is the amount at time 0.

When we are given information of the sort shown in the examples, it is easy to formulate a corresponding equation in the form $A(t) = A_0 b^{t/d}$. It is also easy to convert such an equation into a form having base *e*. We know that the initial value is A_0, and we can compute the relative growth rate as $k = A'(t) \div A(t)$. In fact, since this is a constant, we can compute it at $t = 0$, $k = A'(0) \div A(0) = A'(0) \div A_0$. Differentiating, we find

$$A'(t) = \left(A_0 b^{\frac{t}{d}}\right)' = A_0 \ln(b) b^{\frac{t}{d}} \cdot \frac{1}{d}$$

so

$$A'(0) = A_0 \ln(b) \frac{1}{d}\ .$$

Therefore $k = \ln(b)/d$, leading to $A(t) = A_0 e^{[\ln(b)/d]\cdot t}$. This same result can be found by substituting $e^{\ln(b)}$ for b in the equation $A(t) = A_0 b^{t/d}$.