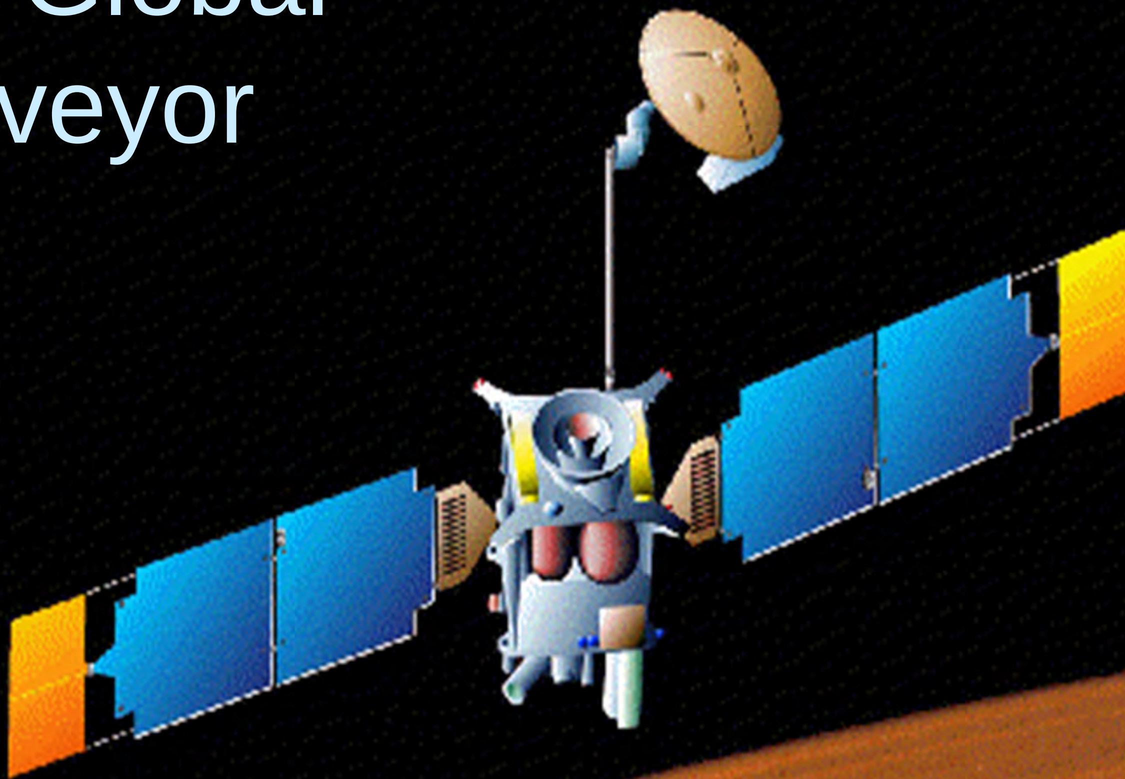
Using Math
to
Predict the Future

Differential Equation Models

Overview

- Observe systems, model relationships, predict future evolution
- One KEY tool: Differential Equations
- Most significant application of calculus
- Remarkably effective
- Unbelievably effective in some applications
- Some systems inherently unpredictable
- Weather
- Chaos

Mars Global Surveyor

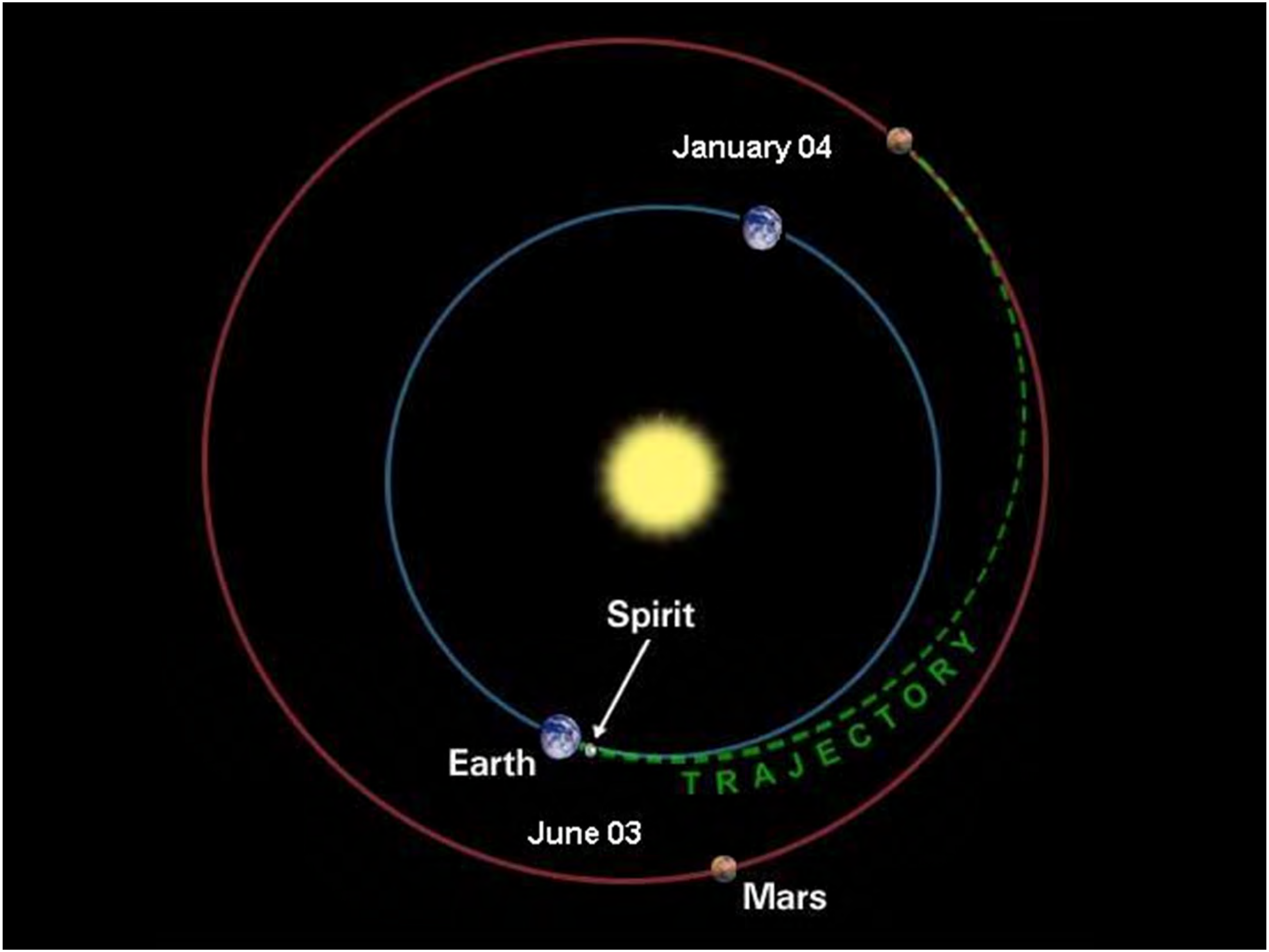


- Launched 11/7/96
- 10 month, 435 million mile trip
- Final 22 minute rocket firing
- Stable orbit around Mars

Mars Rover Missions

- 7 month, 320 million mile trip
- 3 stage launch program

- Exit Earth orbit at 23,000 mph
- 3 trajectory corrections en route
- Final destination: soft landing on Mars


January 04
Spirit
Earth
TRAJECTORY
June 03
Mars

Interplanetary Golf

- Comparable shot in miniature golf
- 14,000 miles to the pin
  more than half way around the equator
- Uphill all the way
- Hit a moving target
- T off from a spinning merry-go-round

Course Corrections

- 3 corrections in cruise phase
- Location measurements
- Radio Ranging to Earth
  Accurate to 30 feet
- Reference to sun and stars
- Position accurate to 1 part in 200 million -- 99.9999995% accurate

How is this possible?

One word answer:

Differential Equations

(OK, 2 words, so sue me)

Reductionism

- Highly simplified crude approximation
- Refine to microscopic scale
- In the limit, answer is exactly right
- Right in a theoretical sense
- Practical Significance: highly effective means for constructing and refining mathematical models

Tank Model Example

- 100 gal water tank
- Initial Condition: 5 pounds of salt dissolved in water
- Inflow: pure water 10 gal per minute
- Outflow: mixture, 10 gal per minute
- Problem: model the amount of salt in the tank as a function of time

In one minute …

- Start with 5 pounds of salt in the water
- 10 gals of the mixture flows out
- That is 1/10 of the tank
- Lose 1/10 of the salt or .5 pounds
- Change in amount of salt is –(.1)5 pounds
- Summary: $\Delta t = 1$, $\Delta s = -(.1)(5)$

Critique

- Water flows in and out of the tank continuously, mixing in the process
- During the minute in question, the amount of salt in the tank will vary
- Water flowing out at the end of the minute is less salty than water flowing out at the start
- Total amount of salt that is removed will be less than .5 pounds

Improvement: ½ minute

- In .5 minutes, water flow is .5(10) = 5 gals
- IOW: in .5 minutes replace .5(1/10) of the tank
- Lose .5(1/10)(5 pounds) of salt
- Summary: Δt = .5, Δs = -.5(.1)(5)

- This is still approximate, but better

Improvement: .01 minute

- In .01 minutes, water flow is .01(10) = .01(1/10) of full tank
- IOW: in .01 minutes replace .01(1/10) = .001 of the tank
- Lose .01(1/10)(5 pounds) of salt
- Summary: $\Delta t = .01$, $\Delta s = -.01(.1)(5)$

- This is still approximate, but even better

Summarize results

Δt (minutes)	Δs (pounds)
1	-1(.1)(5)
.5	-.5(.1)(5)
.01	-.01(.1)(5)

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1	-1(.1)(5)
.5	-.5(.1)(5)
.01	-.01(.1)(5)
h	-h(.1)(5)

$$\frac{\Delta s}{\Delta t} \cong -.1(5) \xrightarrow{\text{Limit}} \frac{ds}{dt} = -.1(5)$$

Other Times

- So far, everything is at time 0
- s = 5 pounds at that time
- What about another time?
- Redo the analysis assuming 3 pounds of salt in the tank
- Final conclusion:

$$\frac{ds}{dt} = -.1(3)$$

So at any time…

If the amount of salt is s,

$$\frac{ds}{dt} = -.1(s)$$

We still don't know a formula for $s(t)$

But we do know that this unknown function must be related to its own derivative in a particular way.

Differential Equation

- Function $s(t)$ is unknown
- It must satisfy $s'(t) = -.1\, s(t)$
- Also know $s(0) = 5$
- That is enough information to completely determine the function:

$$s(t) = 5e^{-.1t}$$

Derivation

- Want an unknown function $s(t)$ with the property that $s'(t) = -.1\, s(t)$
- Reformulation: $s'(t) / s(t) = -.1$
- Remember that pattern – the derivative divided by the function?
- $(\ln s(\mathrm{t}))' = -.1$
- $(\ln s(\mathrm{t})) = -.1\, t + C$
- $s(t) = e^{-.1t + C} = e^{C} e^{-.1t} = Ae^{-.1t}$
- $s(0) = Ae^{0} = A$
- Also know $s(0) = 5$. So $s(t) = 5e^{-.1t}$

Relative Growth Rate

- In tank model, $s'(t) / s(t) = -.1$
- In general $f'(t) / f(t)$ is called the **relative growth rate** of *f* .
- AKA the percent growth rate – gives rate of growth as a percentage
- In tank model, relative growth rate is constant
- Constant relative growth *r* always leads to an exponential function Ae^{rt}
- In section 11.5, this is used to model population growth

Required Knowledge

to set up and solve differential equations

- Basic concepts of derivative as instantaneous rate of change
- Conceptual or physical model for how something changes over time
- Detailed knowledge of patterns of derivatives

Applications of Tank Model

- Other substances than salt
- Incorporate additions as well as reductions of the substance over time
- Pollutants in a lake
- Chemical reactions
- Metabolization of medications
- Heat flow

Miraculous!

- Start with simple yet plausible model
- Refine through limit concept to an exact equation about derivative
- Obtain an exact prediction of the function for all time
- This method has been found over years of application to work incredibly, impossibly well

On the other hand…

- In some applications the method does not seem to work at all
- We now know that the *form* of the differential equation matters a great deal
- For certain forms of equation, theoretical models can never give accurate predictions of reality
- The study of when this occurs and what (if anything) to do is part of the subject of CHAOS.