Calculus 1

Some Examples of Differential Equations

In our earlier discussion of differential equations I showed how simple models for time varying quantities can lead to differential equations. For easy reference, several of these are shown below. In addition, I have included a few other examples that are developed either in our text book or in other sources.

- Tank Model. Assume that a tank contains water with a disolved substance (such as sugar), and that water flows into and out of the tank at a constant rate. The inflow is pure water. It mixes with the solution in the tank, and the mixture flows out. This leads to a differential equation of the following form:

  $$\frac{dA}{dt} = -.1A$$

  where A is the amount of dissolved substance at time t. The constant (-.1) derives from assumptions about the rate of flow. In particular, it indicates that in our model the inflow and outflow rates correspond to one tenth of the volume of the tank in each unit of time.

- Revised Tank Model. This is like the first example, but now we assume that the inflow is not pure. Rather, it has a fixed concentration of the dissolved substance, so that in each unit of time 250 grams of the substance are added to the tank. The differential equation is

  $$\frac{dA}{dt} = -.1A + 250.$$

- Newton Cooling. A warm object (such as a cup of coffee) sits in a room at a lower temperature. We are interested in how the temperature of the coffee changes over time. If the temperature in the room is maintained at 70 degrees, and if $H(t)$ is the temperature of the coffee at time t, the differential equation is of the form

  $$\frac{dH}{dt} = -.3(H - 70)$$

  where the constant (.3) is a parameter that depends on how rapidly heat is transferred from the coffee to the surroundings. If the cup is well insulated this constant will be lower; if the cup conducts heat easily the constant will be higher.

- Snowplow Problem. A snowplow is operating in a heavy blizzard. As the snow gets deeper, the snowplow slows down. We want to understand how the snowplow's rate of progress changes over time. In this model, we let $S(t)$ be the length of road that has been plowed during t hours. We also assume the snowplow began operating at time $t = 0$ so that $S(0) = 0$. That is, at the start, the snowplow had cleared 0 miles of road. Under some simple assumptions about how the snow falls and how the plow operates, a differential equation of the following form is derived:

  $$\frac{dS}{dt} = \frac{1,000,000}{5280(10)(1 + .25t)}.$$

This equation includes five constants: 1,000,000, 5280, 10, 1, and .25. These arise in the model in connection with the following assumptions: the plow can clear 1,000,000 cubic feet of snow per hour; there are 5280 feet in a mile; the snowplow blade is 10 feet wide, when the plow started the snow was 1 foot deep, and the snow is falling at a rate of .25 inches per hour.

- Exponential population growth. The differential equation

$$\frac{dP}{dt} = kP$$

  says that the rate of change of population is proportional to the population size P. The value of k is determined by some measurement of the overall growth rate, such as doubling time.

- Logistic Growth. This is a variant of exponential growth, with an equation of the form

$$\frac{dP}{dt} = m(L - P)P.$$

  Comparing this equation to the preceding one, we see that the constant k has been replaced with $m(L - P)$. This takes into account the idea that the rate of population growth must diminish as the population gets larger due to competition for limited resources. A specific example might look like this:

$$\frac{dP}{dt} = .01(25 - P)P.$$

  For this example, we assume that P is in units of thousands, and the constant 25 indicates that a population of 25,000 will be in equilibrium with its environment. The other constant, .01, indicates that when the population is small it grows approximately exponentially with a relative growth rate of $.25 = (.01)(25)$.

- Torricelli's Law. This states that the rate at which a fluid flows from a drain at the bottom of a tank is proportional to the squareroot of the depth of the fluid in the tank. When the tank is full the fluid is deep and it flows out rapidly. When the tank is nearly empty the fluid will be shallow and the flow is much slower. In applying this idea, the shape of the tank has an impact. A conical tank will drain differently than a cylindrical one, because the depth decreases differently in the two cases. As an example, consider a conical tank 30 feet tall and with a radius of 20 feet at the top. If the rate at which the fluid flows out is k times the squareroot of the depth of the fluid in the tank, then Torricelli's law leads to the following differential equation for the depth of the water, $h(t)$:

$$\frac{4\pi}{9}h^2\frac{dh}{dt} = -k\sqrt{h}.$$