Elementary Math Models
Spring 2015
Exam 1 Name ____________________

The following formulas might be useful on this exam.

$$a_{n+1} = a_n + d \qquad y = mx + b \qquad (y - y_0) = m(x - x_0)$$

$$a_n = a_0 + dn \qquad \frac{x}{a} + \frac{y}{b} = 1$$

Instructions: Read the questions carefully, and be sure to answer all parts of each question. For full credit on non-essay questions, you must show work or give some explanation of your method. You must communicate to me **how** you reached your answer.

1. [**25 points**] Give brief definitions for each of the following terms (this page and the next). Each answer should explain to someone who has never studied the material in this course what each term **is**. Also, each answer can *include* an example, but should be more than just an example.

a. recursion or recursive method

Recursion is a repetitive process for generating terms of a sequence by operating on one or more preceding terms of the sequence, possibly also using constants and position numbers . Producing the terms of a sequence in this way is a recursive method.

b. position number

Position numbers are consecutive whole numbers associated with terms of a sequence to indicate the position of each term within the sequence. For example, in the sequence 2, 4, 6, 8, ... if we think of 2 as being in position 1, then 4 is in position 2, 6 is in position 3, and so on.

c. functional equation for a sequence

A functional equation expresses the general term of a sequence as a function of the position number. For example, consider the sequence 2, 4, 6, 8, ... where we take 2 as being in position 1, 4 in position 2, and so on. A functional equation for this sequence is then given by $a_n = 2n$, because $a_1 = 2 = 2 \cdot 1$, $a_2 = 4 = 2 \cdot 2$, $a_3 = 6 = 2 \cdot 3$, and so on. Note that the functional equation allows us to determine any term of the sequence directly, without having to first compute any other terms. Thus, using the equation we can compute $a_{50} = 2 \cdot 50 = 100$ without finding any other terms.

d. arithmetic growth

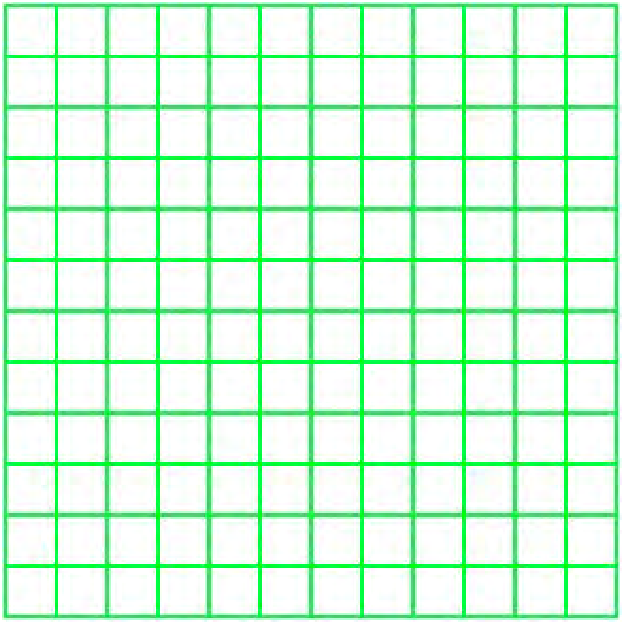
Arithmetic growth is a pattern in a number sequence in which each term differs from the preceding term by a constant amount. For example, the sequence 8, 11, 14, 17, 20, ... is an instance of arithmetic growth because each term is three more than the preceding term.

e. slope of a line

This is from section 2.3 and is not covered on our exam 1.

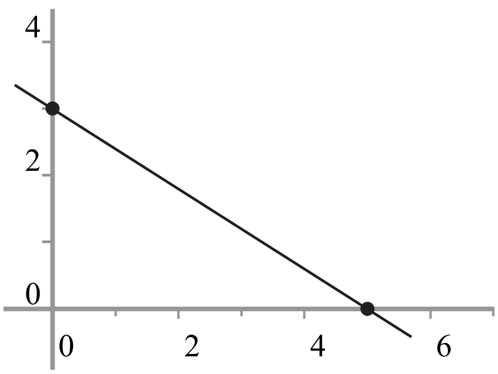
2. Math Skills **[30 points]** (This page and the next two)

a. On the graph below draw a straight line that goes through the point $(-4, 1)$ and has a slope of $5/4$. Draw in and label your axes. To the right of the graph, give the equation of the line.



The items on this page are from section 2.3 and are not covered on our exam 1.

b. Find an equation for the line shown in the graph below. Show your work or explain your method.



c. A number pattern is given by: $3, 10, 17, 24, 31, \cdots$. Denote the terms of this pattern using subscript notation, as follows: $a_0 = 3$, $a_1 = 10$, and so on. Find a_5 and a_{1000}. Show work or explain how you found your answers.

Each term in this sequence is 7 more than the preceding term, so this is an instance of arithmetic growth. Notice that 31 is the term a_4 so 31+7=38 is the next term, and that is a_5. To find a_{1000}, we use the functional equation for the sequence, given by $a_n = 3+7n$. Thus, $a_{1000} = 3 + 7 \cdot 1000 = 7003$.

d. A number pattern starts with $a_0 = 10$ and obeys the difference equation $a_{n+1} = a_n + 5 - n$. Find a_1 and a_2. Show work or explain how you found your answers.

Substitute 0 for every *n* in the difference equation, to find

$$a_{0+1} = a_0 + 5 - 0 = 10 + 5 - 0 = 15.$$

This shows that

$$a_1 = 15.$$

Repeat the same process, but substitute 1 for every *n*. That produces

$$a_{1+1} = a_1 + 5 - 1 = 15 + 5 - 1 = 19.$$

That is, $a_2 = 19$.

e. Solve this equation for x: $500 - 28x = 150$. Show all your steps, and verify that your answer is correct.

500 - 28*x* = 150

- 28*x* = 150 - 500 = -350

x = (-350)/(-28) = 50/4 = 25/2

Checking the answer: 500 - 28(25/2) = 500 - 28(12.5) = 150, which shows that the answer is correct.

3. [**30 points**] A county board of education has been gathering data on home schooling for several years. The data show that in 2007 there were about 3200 home schooling families, in 2008 there were 4500, and in 2009 there were 5800. Make up an arithmetic growth model for this situation. Include the following in your answer:

- A data table
- Definitions for your variables
- A difference equation for your model
- A functional equation for your model
- A graph for your model (use the grid provided on page 7)
- A prediction, based on the model, of the number of home schooling families in 2018 (plus the work that justifies this prediction)
- A prediction, based on the model, of the year in which the number of home schooling families will reach or exceed 50,000 (plus the work that justifies this prediction)

Additional work space is provided on the next page.

We treat the annual figures for number of home schooling families as terms of a sequence: 3200, 4500, 5800, ... These can be included in a data table as shown below at right, where we continue the pattern of the first three terms by increasing each term by 1300 over the previous term.

n	f_n
0	3200
1	4500
2	5800
3	7100
4	8400
5	9700

The variables that appear in the table are n, the position number, and f_n, the terms of the sequence. In the problem context, n is in units of years, with $n = 0$ for 2007, and f_n represents the number of home school families in year n.

The difference equation for this sequence is $f_{n+1} = f_n + 1300$ because each term can be found by adding 1300 to the preceding term. This also shows that the arithmetic growth parameter $d = 1300$ for this sequence.

Functional Equation: Knowing that $d = 1300$ and $f_0 = 3200$, we obtain the functional equation

$$f_n = 3200 + 1300n.$$

Prediction for 2018: Year 0 is 2007, so year 11 is 2018. Thus according to our model, f_{11} represents the number of home schooling families for 2018. Using the functional equation, we find

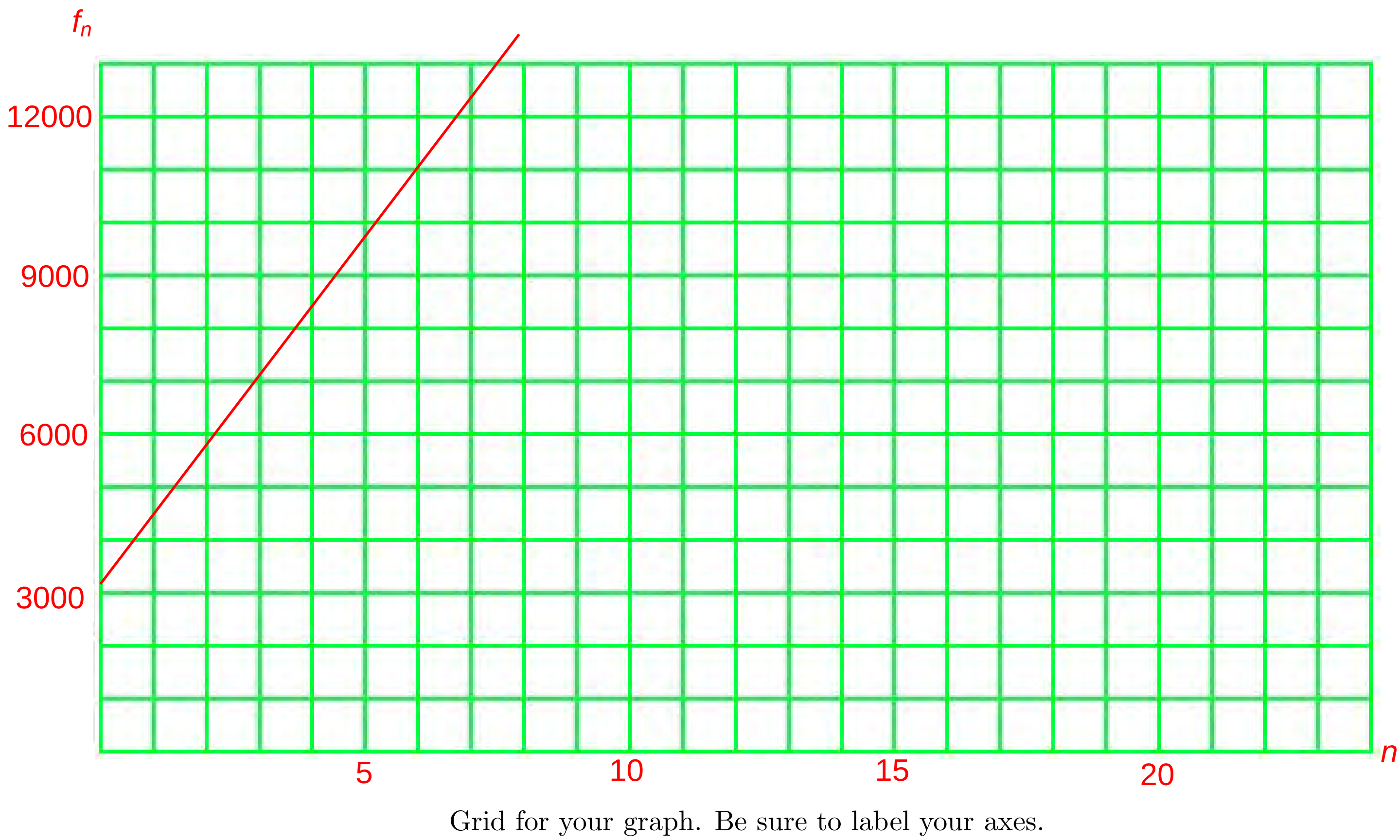
$$f_{11} = 3200 + 11 \cdot 1300 = 17500.$$

Prediction for when there are 50,000 or more home schooling families: We want to know for what n will f_n be 50,000 or more. So we substitute 50,000 for f_n in the functional equation, and solve for n, as shown below.

$$50000 = 3200 + 1300n$$
$$50000 - 3200 = 1300n$$
$$46800 = 1300n$$
$$n = 46800/1300 = 36.$$

So the number of home schooling families is predicted by the model to reach 50000 in year 36, which corresponds to 2007+36 = 2043.

Additional workspace for problem 3.



Grid for your graph. Be sure to label your axes.

Part II. Multiple Choice. [15 points] Circle the correct response for each item. You may write a brief explanation if you wish, but are not required to do so.

1. Which verbal description fits the following sequence? $0, 1, 3, 6, 10, 15, 21, \cdots$.
   (Assume 1 is the position number for the first term.)

   a. These terms are multiples of three.

   b. If you double any term and then add one, you'll get the next term.

   c. If you add the position number to any term, you'll get the next term.

   d. This is arithmetic growth with a starting term 0 and an added amount of 3 at each step.

2. Which equation fits the sequence described below?

   Three times the previous term plus 2 produces the next term.

   a. $a_n = 3n + 2$ b. $a_n = 3a_{n-1} + 2$ c. $a_{n+1} = 3 + 2a_n$ d. $a_{n+1} = 3a_{n+2}$

3. For a sequence $9, 3, 15, 23, 19, 17, \cdots$, if $a_n = 3$ then

   a. $a_{n+1} = 15$ b. $a_{n+1} = 23$ c. $a_{n+1} = 4$ d. $a_{n+1} = 16$

4. Which of the following would **not** indicate arithmetic growth of a number sequence?

   a. When plotted on a graph, the data appear to lie on a straight line.

   b. The terms are given by a functional equation of the form $a_n = a_0 + dn$.

   c. Increases between successive terms follow the pattern of even numbers $2, 4, 6, 8, \cdots$.

   d. The terms can be generated recursively by repeatedly adding a specific constant.

5. Which of the following is most correct?

   a. An arithmetic growth sequence should be used in a model only if a graph of the data shows points that fall exactly on a straight line.

   b. An arithmetic growth sequence may be used if the differences between data points are approximately constant.

   c. Arithmetic growth sequences are rarely used in real applications.

   d. In arithmetic growth models, proportional reasoning should not be assumed.