Elementary Math Models

Sample Exam 2

Name ______________________________

Part I. Multiple Choice. [24 points] Circle the correct response for each item. You may write a brief explanation if you wish, but are not required to do so.

1. An epidemiologist is developing a model for the spread of measles during a recent outbreak in the US. In the model M_n is the number of people who have been infected with measles by the end of week n, and the following functional equation is found:

$$M_n = 10,000 + 846n.$$

Which statement most correctly describes this equation?

   a. It expresses M_n as a function of n.
   b. It expresses n as a function of M_n.
   c. It expresses M_n as a quadratic function of n.
   d. It is an example of the quadratic formula.

2. A quadratic growth sequence has constant second differences all equal to 25. If $a_1 = 1000$ and $a_5 = 5000$, which of the following must be true?

   a. $a_9 = 9,000$ b. $a_9 > 9,000$ c. $a_9 < 9,000$ d. Can't tell from the given information.

3. Which of the following is NOT a valid reason for adopting a quadratic growth model?

   a. The differences of the differences form a simple pattern such as $3, 6, 9, 12, 15, 18, \cdots$.
   b. The second differences are nearly, but not exactly, equal.
   c. A recursive analysis shows that the difference between a_n and a_{n+1} is a linear function of n.
   d. We are interested in the running totals of an arithmetic growth sequence.

4. In a model of social media networks, two participants who are "friends" are said to share a friendship link. In the model, the number of friendship links is given as a quadratic function of the number of participants. If the number of participants is doubled in the model, which the of following is most likely to occur?

   a. The number of friendship links will approximately double.
   b. The number of friendship links will exactly double.
   c. The number of friendship links will not change.
   d. The number of friendship links will approximately quadruple (that is, increase by a factor of four).

The following equations might be useful on this exam. $1+2+3+\cdots+n = n(n+1)/2$

$$a_{n+1} = a_n + d \qquad a_{n+1} = a_n + d + en \qquad y = mx + b \qquad \frac{x}{a} + \frac{y}{b} = 1$$

$$a_n = a_0 + dn \qquad a_n = a_0 + dn + e\left(\frac{(n-1)n}{2}\right) \qquad y - y_0 = m(x - x_0)$$

Part II: Free Answer. Read the questions carefully, and be sure to answer all parts of each question. For full credit on non-essay questions, you must show work or give some explanation of your method. You must communicate to me **how** you reached your answer.

1. Essay Questions on Quadratic Growth [**20 points**]. Write short essay responses to the items on this page and the next. Your answers should be written in one or more complete English sentences. Each answer can *include* an example, but there should also be an explanation of the meaning and significance of any examples you use.

a. Explain how to recognize quadratic growth in a number sequence (or number pattern).

Create a data table and compute first and second differences. A pattern of arithmetic growth in the first differences indicates the original sequence is an instance of quadratic growth. Likewise, constant second differences (so that all the second differences are equal) indicates quadratic growth in the original sequence.

b. Is proportional reasoning valid for quadratic growth models? Explain why or why not.

No, proportional reasoning is not valid. Proportional reasoning is valid only for data that create a straight line graph, because proportional reasoning is equivalent to assuming a constant slope in the graph. We know that graphs of quadratic growth sequences are not straight lines, implying that proportional reasoning would not be valid.

c. What is meant by running totals for a number sequence, and what does this topic have to do with quadratic growth?

Given a sequence a_0, a_1, a_2, etc we can create a sequence of sums defined by $s_0 = a_0$, $s_1 = a_0 + a_1$, $s_2 = a_0 + a_1 + a_2$, etc. That is, s_n is the total of all the terms from a_0 through a_n of the original sequence. This sequence of sums is referred to as the running totals for the original sequence. This is relevant in quadratic growth because the running totals for an *arithmetic* growth sequence always form a quadratic growth sequence. This is one of the ways that quadratic growth models arise.

d. In class we discussed the handshake problem, where the king must shake the hand of each guest three times, and each guest has to shake the hand of every other guest once. Let h_n be the total number of handshakes that must occur at an event with the king and n guests. Use a recursive analysis to find a difference equation for h_n, explaining your reasoning.

Suppose that there are 67 guests already at the party, and that all the handshakes have occurred. The number of these handshakes is h_{67}. Now a 68th guest arrives. He or she must shake hands with the king 3 times, and must shake hands once with each of the other 67 guests. So the number of additional handshakes will be 3 + 67. This shows that $h_{68} = h_{67} + 3 + 67$. Similar logic applies with any other number in place of 67. Thus, we see that the sequence h_n obeys the difference equation $h_{n+1} = h_n + 3 + n$.

NOTE: These questions only concern quadratic growth. Also be prepared to show that you understand terms from sections 2.3 and 2.4, such as slope, intercept, and proportional reasoning.

2. Math Skills [**28 points**]. Solve the problems on the this page and the next. You must **show work** for full credit.

a. A number sequence has difference equation $a_{n+1} = a_n + 3 - 2n$ and the starting value is $a_0 = 12$. Use the difference equation to find a_1 and a_2. Show your work.

Substitute 0 for every n in the difference equation. That gives $a_{0+1} = a_0 + 3 - 2(0)$, and so $a_1 = 12 + 3 - 0 = 15$. Now repeat the process, replacing every n with 1. This time we find $a_{1+1} = a_1 + 3 - 2(1)$, and so $a_2 = 15 + 3 - 2 = 16$.

b. A number sequence has difference equation $b_{n+1} = b_n - 3 + 4n$ and a starting value of $b_0 = 24$. Find the functional equation and use it to find b_{20}.

We use the general quadratic growth functional equation replacing b_0 with 24, d with -3 and e with 4. That leads to

$$b_n = 24 - 3n + 4n(n -1)/2$$

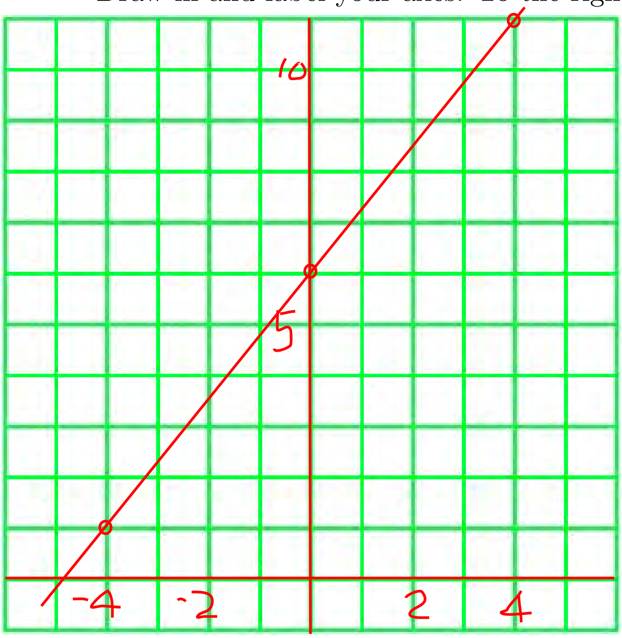
which we can simplify to

$$b_n = 24 - 3n + 2n(n -1).$$

Substituting $n = 20$,

$$b_{20} = 24 - 3(20) + 2(20)(19) = 24 - 60 + 760 = 724.$$

c. On the graph below draw a straight line that goes through the point $(-4, 1)$ and has a slope of $5/4$. Draw in and label your axes. To the right of the graph, give the equation of the line.



As indicated on the graph, the *y* intercept of the line is at 6. Therefore, the equation is *y* = (5/4)*x* + 6. A second approach is to use the point slope equation. That produces

y - 1 = (5/4)(*x* - -4)

or

y - 1 = (5/4)(*x* + 4) .

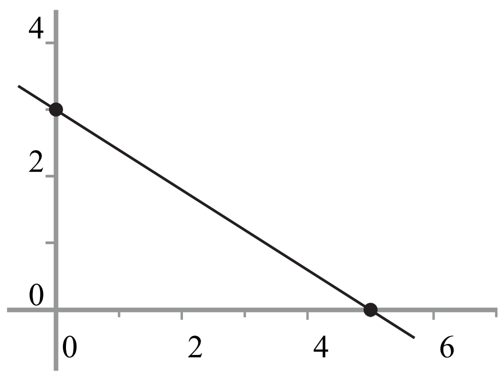
Solving for *y* and distributing the factor of 5/4 leads to

y = (5/4)*x* + 20/4 + 1

or more simply

y = (5/4)*x* + 6.

d. Find an equation for the line shown in the graph below. Show your work or explain your method.



In this graph we see that the *x* intercept is 5 and the *y* intercept is 3. We can therefore immediately use the two intercept equation, producing

x/5 + *y*/3 = 1.

This is a valid equation for the line. However, as a simplification, multiply both sides by 15 and obtain

3*x* + 5*y* = 15.

If desired, we can also solve this equation for *y*, ultimately reaching *y* = (-3/5)*x* + 3. Alternatively, using the two points identified in the graph, compute the slope as -3/5 and use the *y* = *mx* + *b* form. That leads to the same equation as before: *y* = (-3/5)*x* + 3.

4. Modeling Problem [**20 points**] (this page and the next). A research team is studying the growth of traffic in a suburban area. Based on their data and some analysis, they have estimated the average daily number of miles driven by each car in the area each year for the past several years. Here are their data:

Year Number	0	1	2	3	4
Average Miles Driven per day	12.5	15.0	18.0	21.5	25.5

First differences: 2.5, 3, 3.5, 4

Second differences: .5, .5, .5

a. What kind of growth model would be appropriate for this problem? Why?

First and second differences have been added to the table above. The second differences are constant. Therefore, the sequence shown in the second line of the table is a quadratic growth sequence. Therefore, a quadratic growth model would be appropriate here.

b. Develop an appropriate model for the data. Define the variables that you use, and give both a difference equation and a functional equation, clearly labeled.

We consider the sequence M_0, M_1, M_2, etc where M_n equals the average number of miles driven per day during year n. In particular, M_n is in units of miles and n in units of years. As shown above, this is a quadratic growth sequence. We can find the difference equation using the standard quadratic growth difference equation with $d = 2.5$ and $e = .5$. Thus we have

difference equation: $M_{n+1} = M_n + 2.5 + .5n$.

From the table, $M_0 = 12.5$, so we also have the

functional equation: $M_n = 12.5 + 2.5n + .5n(n - 1)/2$.

Simplifying slightly, we can also write

functional equation: $M_n = 12.5 + 2.5n + .25n(n - 1)$.

As a check, we can replace n with 4. The functional equation produces

$M_4 = 12.5 + 2.5(4) + .25(4)(3) = 12.5 + 10 + 3 = 25.5$.

This is the correct value according to the table.

c. According to your model, how many miles will be driven per day, on the average, in year 5?

From the original table, we know that $M_4 = 25.5$, so we can use the difference equation to find

$$M_5 = M_4 + 2.5 + .5(4) = 25.5 + 2.5 + 2 = 30.$$

This checks out with the table, because it gives the next first difference as 4.5, as it should. These results show that on the average, 30 miles will be driven per day in year 5.

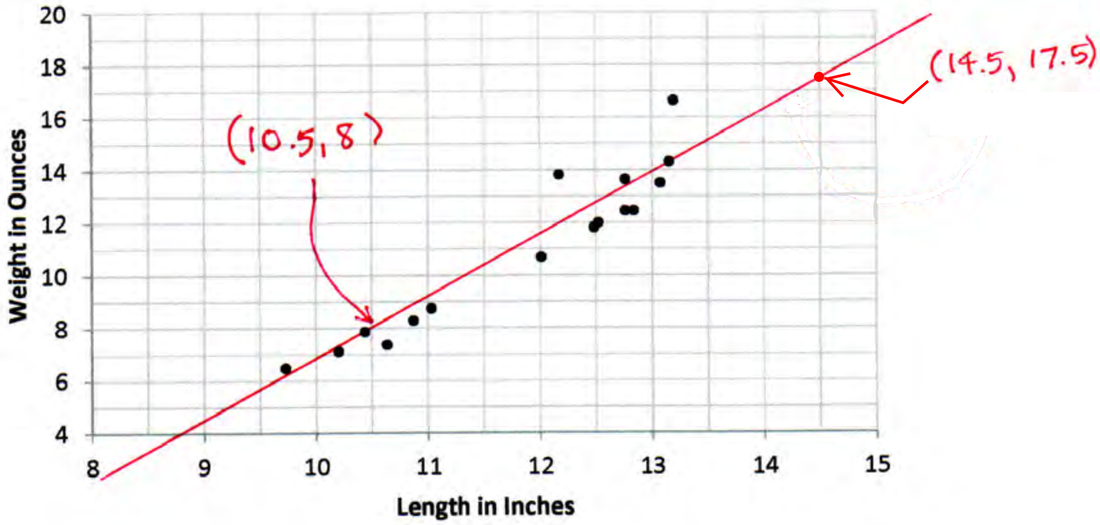
d. According to your model, how many miles will be driven per day, on the average, in year 20?

This time we use the functional equation with $n = 20$. That produces

$$M_{20} = 12.5 + 2.5(20) + .25(20)(19) = 12.5 + 50 + 95 = 157.5.$$

Therefore, according to the model, on the average, 157.5 miles will be driven per day in year 20.

5. [8 points] In the graph below each point represents the weight and length of one fish taken from a population in Washington state.



a. Using a straight edge, draw a straight line that fits the data as closely as possible.

b. Find an equation of the line you drew in part a, expressing weight W as a function of length L. Show work and/or explain your method.

Drawn line goes through (10.5, 8) and (14.5, 17.5) so the slope is

$m = \frac{17.5 - 8}{14.5 - 10.5} = \frac{9.5}{4} = 2.375$. From the point slope formula we

find $(y-8) = 2.375(x-10.5)$ or $y = 8 + 2.375(x-10.5)$. Replacing x with L and y with W produces

$$W = 8 + 2.375(L - 10.5)$$

c. Using your equation as a linear model for the data, about what length would you predict for a fish that weighs 12 ounces?

Set $W = 12$ to find

$$12 = 8 + 2.375(L - 10.5)$$
$$4 = 2.375(L - 10.5)$$
$$\frac{4}{2.375} = L - 10.5$$
$$\frac{4}{2.375} + 10.5 = L$$

Using a calculator, $L = 12.1842\ldots$

So, we can predict a length of approximately 12.2 inches.