Math 155 Finite Mathematics - Elementary Models

Final Exam (Sample) Name ______________________

The following formulas might be useful on this exam.

$b^r b^s = b^{r+s}$ $\quad (b^r)^s = b^{rs}$ $\quad b^r/b^s = b^{r-s}$ $\quad b^{-r} = 1/b^r$

$a_{n+1} = a_n + d$ $\quad a_{n+1} = a_n + d + en$ $\quad a_{n+1} = ra_n$ $\quad a_{n+1} = ra_n + d$

$a_n = a_0 + dn$ $\quad a_n = a_0 + dn + e\left(\frac{(n-1)n}{2}\right)$ $\quad a_n = a_0 r^n$ $\quad a_n = a_0 r^n + d\left(\frac{1-r^n}{1-r}\right) = a_0 r^n + d\left(\frac{r^n-1}{r-1}\right)$

$a_n = E + (a_0 - E)r^n$ where $E = \frac{d}{1-r}$

$y = mx + b$ $\quad y - y_0 = m(x - x_0)$ $\quad \frac{x}{a} + \frac{y}{b} = 1$ $\quad e = 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 3} + \frac{1}{2\cdot 3\cdot 4} + \cdots$

$m = \frac{\text{rise}}{\text{run}}$ $\quad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$ $\quad \frac{-b}{2a}$ $\quad 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

$a(t) = a_0 r^{t/d}$ $\quad b^x = c$ if $x = \frac{\log c}{\log b} = \frac{\ln c}{\ln b}$ $\quad b = e^{\ln b} = 10^{\log b}$ $\quad 1 + r + r^2 + \cdots + r^n = \left(\frac{1-r^{n+1}}{1-r}\right) = \left(\frac{r^{n+1}-1}{r-1}\right)$

a_n remains between 0 and L if $mL < 4$ $\quad$ a_n will go to 0 if $0 \le mL \le 1$ $\quad$ a_n levels off at $L - \frac{1}{m}$ if $1 < mL \le 3$

$p_{n+1} = \frac{p_n}{mp_n + b}$ $\quad p_n = \frac{A}{1+Bb^n}$ where $A = \frac{1-b}{m}$ and $B = \frac{A}{p_0} - 1$

1. Give brief explanations of each of the following terms or expressions (parts d-f are from Section 6.2):

a. logistic growth model
b. fixed point of a model
c. linear growth factor model
d. butterfly effect
e. nonlinear difference equation
f. robustness of a model

Answers:

a. Logistic growth is a modified version of geometric growth. Whereas geometric growth assumes a constant growth factor between successive terms, in logistic growth the growth factor is assumed to decrease as the sequence terms increase. More specifically, it is assumed that the growth factor to go from p_n to p_{n+1} is given by a linear function of p_n.

b. A fixed point of a difference equation model is a value that is unchanged by the recursion. If a_n is equal to a fixed point then a_{n+1} will also equal that same value. If a sequence produced by a difference equation approaches an equilibrium value E, then E must be a fixed point.

c. A linear growh factor model is an equation of the form $r = mp + b$ giving the growth factor r as a linear function of the population size p, with constants m and b representing the slope and intercept of the linear equation.

d. The butterfly effect is the situation where very slight changes in of a model's parameters can lead to major qualitative differences in the way the results produced by the model. We saw an example of this in the reading in a population model for fish. With an initial population of 100,000, the model gave one set of predictions; with an initial population of 100,010, the predictions were completely different (after a number of terms at the start).

e. In general, a difference equation can be expressed in the form of the equation $a_{n+1} = f(a_n)$ or $a_{n+1} = f(a_n, n)$. This says that each term of the sequence is computable by applying some function to the preceding term, or to a combination of the preceding term and the position number. For a *linear* difference equation, the function f is a linear function.

f. Robustness of a model means that the model will produce consistent predictions, even when the parameters are modified. For example, mixed models are robust. If the model approaches an equilibrium with one set of parameters, then it will also approach a slightly different equilibrium after slight changes in the parameters. The point is that we get consistent qualitative predictions even when our estimated parameters are inaccurate.

2. In a factory, water is constantly flowing through a large vat, where it is mixed with a solid chemical. A chemist is studying the way the chemical dissolves and develops a mixed growth model. At the start of the experiment 100 pounds of the chemical in a solid form are added to the vat. Each hour, about one tenth of the chemical dissolves, leaving 9/10 still lying at the bottom of the vat, and another 20 pounds of the solid chemical is added to the vat. Develop a mixed model for this situation. Let s_n be the amount of solid chemical at the bottom of the vat after n hours, starting with $s_0 = 100$. In your answer, include

a. The first few terms: s_1, s_2, and s_3.

b. A difference equation for s_n

c. A functional equation for s_n

d. A discussion of the long range predictions that can be made based on the model.

Answers:

a. Starting with $s_0 = 100$, after one hour 10 pounds have dissolved leaving 90 pounds and another 20 pounds are added, leaving $s_1 = 110$ pounds. After another hour one tenth of the 110 pounds has dissolved. This leaves $.90 \cdot 110 = 99$ pounds, and another 20 pounds are added, so $s_2 = 119$. After another hour, the amount will be $s_3 = .90 \cdot 119 + 20 = 127.1$.

b. The difference equation is $s_{n+1} = .9s_n + 20$.

c. The functional equation can be expressed either as

$$s_n = 100(.9^n) + 20\frac{1 - .9^n}{.1} \ ,$$

or, computing $E = 20/(1 - .9) = 200$ and $a_0 - E = 100 - 200 = -100$, as

$$s_n = 200 - 100(.9^n).$$

d. This is a mixed model with $r = .9 < 1$, so we can be confident that the model will approach the equilibrium value $E = 200$. Also, because the initial term is below the equilibrium value, the terms will increase toward 200 from below. Thus, as we proceed further and further forward in the sequence, we will observe terms that increase more and more closely toward 200. These conclusions are confirmed by the simplified functional equation $s_n = 200 - 100(.9^n)$ which we recognize as a negative exponential decay, shifted vertically by 200 units. Therefore, the graph will increase from the initial value 20, gradually leveling off as s_n gets ever closer to 200.

3. For all of the parts of this problem $a_{n+1} = .025(100 - a_n)a_n$.

a. If $a_0 = 10$, find a_1, a_2, a_3, and a_4

Answer:

$$\begin{aligned} a_1 &= .025(100-10)10 = 22.5 \\ a_2 &= .025(100-22.5)22.5 = 43.59375 \\ a_3 &= .025(100-43.59375)43.59375 = 61.474 \\ a_4 &= .025(100-61.474)61.474 = 59.209 \end{aligned}$$

b. Predict approximate values of a_n for large n, say for a_{50} or a_{100}. Justify your answer.

Answer: For this problem $mL = .025 * 100 = 2.5$. That is between 1 and 3 so we can predict that in the long run, the model will approach an equilibrium value given by $L - 1/m = 100 - 1/.025 = 60$.

c. Find a positive fixed point for this model. That is, find a positive number for a_n that leads to exactly the same number for a_{n+1}

Answer: The fixed point is 60, as computed above. It can also be found by solving the equation $.025(100 - a_n) = 1$ which says that the growth factor will be 1 for that particular a_n. To verify that 60 is a fixed point, apply the recursion to 60: $.025(100 - 60)60 = 60$. This shows that if the population is 60, it will stay 60.

4. A logistic growth model for a population has the form $p_{n+1} = m(200 - p_n)p_n$.

a. If $m = .025$ is this a reasonable equation for a population model? Justify your answer.

Answer: For this problem $mL = .025 \cdot 200 = 5$. This is greater than 4, so the model can produce negative population values in some situations. For example, if the population is 100 one time, then the next time it will be $.025(100)100 = 250$. That is above 200 and the next value produced by the model will be negative. So this is not a reasonable model for population growth.

b. If $m = .012$ does the population approach an equilibrium value? If so, what will that equilibrium value be?

Answer: Now $mL = .012 \cdot 200 = 2.4$ which is between 1 and 3, so the population will approach an equilibrium equal to $L - 1/m = 200 - 1/.012 = 116.67$.

c. If $m = .004$ does the population approach an equilibrium value? If so, what will that equilibrium value be?

Answer: With $mL = .004 \cdot 200 = .8$ which is less than 1, the population will approach an equilibrium of 0.

d. If m is some other value, is it possible for the model to become chaotic? If so, under what circumstances? (This is from chapter 14.)

Answer: In order to observe chaos, mL must be larger than 3.5, approximately, (but less than 4) so changing .025 to something between .0175 and .02 could make the model chaotic.

5. A scientist is studying a certain type of plant virus and how it spreads through fields of wheat. In one experiment, there were initially 100 infected plants, and in one week that number had doubled to 200 infected plants (a growth factor of 2). In another experiment the scientist started with 600 infected plants, and after a week that number had grown to 900 infected plants (a growth factor of 1.5.) Based on this information develop a linear equation in which the variables are p, the initial number of infected plants, and r, the growth factor for infected plants over the next week. Then use this equation to develop a logistic growth model for the number of infected plants week by week as an infection spreads.

Answer: Let r be the growth factor for various population sizes given by variable p. The given information can be expressed in table form as follows:

p	r
100	2.0
600	1.5

Create a graph for this with p on the x axis, with r on the y axis, and with two points, (100,2) and (600, 1.5). For the equation of the line we need the slope between the two points, given by $m = \frac{1.5-2}{600-100} = \frac{-.5}{500} = -.001$. The equation of the line is then $(y-2) = -.001(x-100)$, or using the variables r and p, it is $(r-2) = -.001(p-100)$. Algebra can be used to put this in the simpler form $r = 2.1 - .001p$

The logistic growth model based on the preceding equation is

$$p_{n+1} = (2.1 - .001p_n)p_n$$

or, in the more familiar form

$$p_{n+1} = .001(2100 - p_n)p_n$$

6. Develop a *refined logistic growth* model for the situation described in the preceding problem. Find both the difference and functional equations for the model.

Answer: We proceed in a similar fashion, but this time we develop a linear model for the reciprocal of the growth factor, that is, for $1/r$. The first step is to modify our table showing values of p and $1/r$:

p	$1/r$
100	1/2
600	2/3

Create a graph for this with p on the x axis, with $1/r$ on the y axis, and with two points, (100,1/2) and (600, 2/3). For the equation of the line we need the slope between the two points, given by $m = \frac{2/3-1/2}{600-100} = \frac{1/6}{500} = 1/3000$. The equation of the line is then $(y-1/2) = (1/3000)(x-100)$, or using the variables $1/r$ and p, it is $(1/r - 1/2) = (1/3000)(p-100)$. Algebra can be used to put this in the simpler form $1/r = (1/3000)p + 14/30$. Therefore we find the equation for r is

$$r = \frac{1}{(1/3000)p + 14/30}.$$

Multiplying the numerator and denominator by 3000 simplifies a bit, giving

$$r = \frac{3000}{p + 1400}.$$

The refined logistic growth model based on the preceding equation is

$$p_{n+1} = \frac{p_n}{(1/3000)p_n + 14/30}$$

or, in the simplified form

$$p_{n+1} = \frac{3000p_n}{p_n + 1400}.$$

For the functional equation, we use the parameters $m = 1/3000$ and $b = 14/30 = 7/15$. The equilibrium value is $A = (1 - b)/m = (8/15)/(1/3000) = 1600$. We are not given an initial population size, so let us just assume $p_0 = 100$. Then the parameter $B = A/p_0 - 1 = 1600/100 - 1 = 15$. This gives the functional equation as

$$p_n = \frac{1600}{1 + 15 \cdot (7/15)^n}.$$

7. For each situation below, indicate whether the problem should be modeled using arithmetic growth, quadratic growth, geometric growth, mixed growth, logistic growth, or none of these.

a. A population is growing by increasing amounts each year. The first year it grows by 1000, the next year it increases by 1300, the year after that the increase is 1600, then 1900, then 2200, then 2500, and so on.

   Answer: This is quadratic growth: the second differences are all 300.

b. A biologist is working on a biological filter for purifying water. Each time the water is passed through the filter, 87 percent of the inpurities are removed. That is, when you pass the water through the filter the first time, 87 percent of the impurities are removed. Then, if you pass the water through a second time, 87 percent of the remaining impurities are removed, and so on. The model should describe how the total amount of impurities shrinks as the water is put throught the filter over and over again.

   Answer: This is geometric growth: the amount of impurities is reduced by a fixed percentage each time – namely, 13%.

c. A financial planner is managing a trust fund for a college student. The trust fund grows by about 9 percent each year as a result of interest and dividends. The manager pays out $25000 per year to the student for tuition and living expenses. The model should describe how the total amount in the fund goes up or down year by year.

   Answer: This is a mixed model. Each year the fund grows by 9 percent – and that is a growth factor of 1.09, and then there is also a constant subtraction of $25000. So there is both a multiplication by a constant 1.09 and a subtraction of a constant 25000. That is a mixed model.

8. The following is given: $a_{n+1} = (a_n - 2)^2$ and $a_1 = 5$. A student is asked to figure out a_4, and makes the following computations:

$$\begin{aligned} a_2 &= (a_1 - 2)^2 &= (5-2)^2 &= 9 \\ a_3 &= (a_2 - 2)^2 &= (9-2)^2 &= 49 \\ a_4 &= (a_3 - 2)^2 &= (49-2)^2 &= 2,209 \end{aligned}$$

Is this student using a difference equation or a functional equation? Is the method recursive? Explain.

Answer: This is a difference equation, and a recursive method, because each a_n is found using the value of the preceding a_n.

9. The following set of data came up in a real experiment:

Observation Number (n)	Data Value (a_n)
0	0
1	242
2	527
3	854
4	1220
5	1626
6	2072
7	2558

Indicate whether or not each of the following model types would be a good choice for modeling this data set. For each type, either say why it is not a good model type, or give your best possible difference equation using that type of model for the given data.

Answer: This is really only closely modeled by quadratic growth. The first differences are definitely not all the same, but the second differences are 43, 42, 39, 40, 40, 40, and those are all pretty close together, especially the last four. If you divide successive values in the original data, they do not show a similar pattern of nearly equal values, so geometric growth is not really appropriate. Similarly, if you divide successive values among the first differences you again do not observe constant values, so a mixed model would also not be very appropriate.

To find a difference equation for the quadratic model, we need values for the parameters d and e. For d use the first of the first-differences from the data table, in other words $d = 242$. For e, the value is supposed to be the constant second differences. In the data, since the second differences are only approximately constant, we can take a representative value. Something like 41 or 42 works pretty well. So a possible difference equation for the quadratic model would be $a_{n+1} = a_n + 242 + 41n$.

10. A biologist is studying the growth of a population of sea otters. She takes a count of the number of breeding pairs each year, and finds the following data:

Year	1	2	3	4	5	6	7
Pairs	15	20	35	55	90	145	235

The biologist notices the following pattern: From the third year on, the number of breeding pairs for a year is the total of the number of pairs for the two preceding years.

a. According to this pattern, what would you predict for years 8 and 9?

Answer: For year 8 it should be $145 + 235 = 380$, and for year 9 it would be $235 + 380 = 615$.

b. Write a difference equation for this pattern. Define what each variable stands for.

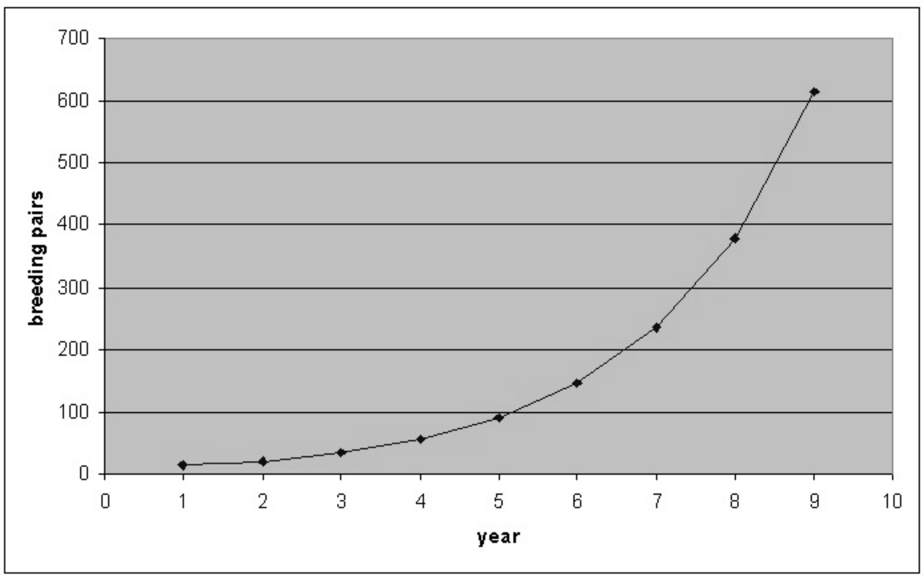
Answer: Let n be the year, and let p_n be the number of pairs observed in year n. The difference equation is then

$$p_{n+2} = p_{n+1} + p_n$$

Another equally valid version is

$$p_{n+1} = p_n + p_{n-1}$$

c. Make a graph for this population model.



d. Would proportional reasoning be reasonable for this model? Explain.

Answer: Proportional reasoning would NOT be appropriate. The graph shows that the model definitely does not follow a straight line and that shows that proportional reasoning would not be valid. Or, looking at the data, in year 1 the value is 15, in year 3 it is 35. Proportional reasoning would lead you to expect the value for year 2, half way between, should be 25. This is wrong, the value in year 2 was 20. Similarly, with an increase of 20 from year 1 to year 3, proportional reasoning would lead you to predict another increase of 20 from year 3 to year 5. This means the value for year 5 should be 55, but instead it is 90, nearly twice what proportional reasoning would predict.