# Extra Problems

2.2.x1 The logical condition *P is true or Q is true but not both* can be expressed symbolically as

Statement A: $(P \vee Q) \wedge \neg(P \wedge Q)$.

Show that this is logically equivalent to

Statement B: $(P \wedge \neg Q) \vee (Q \wedge \neg P)$

by constructing a truth table for each statement and comparing the results. Then derive the equivalence $(P \vee Q) \wedge \neg(P \wedge Q) \equiv (P \wedge \neg Q) \vee (Q \wedge \neg P)$ by algebraically manipulating statement A into statement **B** using rules of logic provided in Theorem 2.8. Finally, explain in words why statement B means the same things as *P is true or Q is true but not both*.

3.4.x1. Without using calculus, prove that

$$\frac{2^n}{100} + \frac{100}{n} > 15$$

for all natural numbers $n$. [Hint: The expression on the left is the sum of two positive quantities. Use a case argument where one case considers when $2^n / 100 > 15$, a second case considers where $100/n > 15$, and the remaining case is dealt with by direct computation.

4.1.x1 This exercise concerns 2 by 2 matrices. Matrices are rectangular tables of numbers. They are a key topic in linear algebra and occur in many other courses. As a particular case, a 2 by 2 matrix is a table of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where the entries $a$, $b$, $c$, and $d$ are specific numbers. Such matrices can be multiplied by the rule $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}$. Each entry in the result on the right is a combination of a row from the left factor and a column from the right factor. For example, we find the upper left entry $aw+by$ by combining $[a \quad b]$ and $\begin{bmatrix} w \\ y \end{bmatrix}$. Specifically, we multiply the corresponding entries of $[a \quad b]$ and $\begin{bmatrix} w \\ y \end{bmatrix}$ and then add.

For this exercise, let $A$ be the specific matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. We can repeatedly multiply $A$ by itself to calculate *powers* of $A$. Thus $A^2 = A{\cdot}A$, $A^3 = A^2{\cdot}A$, $A^4 = A^3{\cdot}A$, and so on.

a. Compute $A^2$, $A^3$, $A^4$, $A^5$. Based on the results, conjecture a formula for $A^n$.
b. Use induction to prove your formula is valid for all natural numbers $n$.

A standard topic in calculus 2 is Taylor Series. If $f(x)$ is an infinitely differentiable function at $x = a$, the Taylor series for $f$ at $a$ is defined as $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$. (The expression $f^{(n)}(a)$ means the $n^{\text{th}}$ derivative of $f$ evaluated at $x = a$.) In using Taylor series, it is often helpful to find a formula for $f^{(n)}(a)$. The next two problems apply proof by induction in this context.

4.1.x2 Let $f(x) = xe^x$.

a. Find $f'(x)$, $f''(x)$, $f'''(x)$. Based on your results, conjecture a formula for $f^{(n)}(x)$.
b. Use induction to prove your formula is valid for all natural numbers $n$.

4.1.x3 Let $f(x) = \ln x$ (for $x > 0$). To formulate the Taylor series for $f$ at $a = 1$, it is necessary to find the coefficients $\frac{f^{(n)}(1)}{n!}$ for any natural number $n$.

a. Compute the values of $f^{(n)}(x)$ for $n = 1, 2, 3$, and 4, and conjecture a formula for any natural number $n$. Use induction to prove your formula is valid. [Hint: your formula can be simply expressed using factorials.]
b. Prove your formula in part *a*.
c. Use your formula in part *a* to find an equation for $\frac{f^{(n)}(1)}{n!}$.
d. What is the Taylor series for $\ln x$ at $x = 1$?

4.1.x4 Let $f(x) = \frac{1}{1+x}$. What happens if we start with $x = 1$, and then repeatedly apply $f$? That is, we start with 1, apply $f$ to find 1/2, then apply $f$ to *that* and find $\frac{1}{1+\frac{1}{2}} = \frac{2}{3}$. Next we apply $f$ to 2/3 to find $\frac{1}{1+\frac{2}{3}} = \frac{3}{5}$. Continue this process for several more steps and try to find a pattern. Use induction to prove that your pattern persists for all natural numbers $n$. [Express each result as a simple fraction in lowest terms. What familiar numbers appear in the numerators and denominators?]

4.1.x5 The $n$ function product rule. We know from calculus that if $f(x)$ and $g(x)$ are differentiable, then so is their product, with derivative given by $f'(x)g(x)+f(x)g'(x)$. We can use this to find rules for 3, 4, or any other number of differentiable functions. For example, we can consider $fgh$ to be given by $(fg)h$, and use the product rule twice. Applying this approach, propose a product rule for $n$ functions, and prove that the rule is valid using induction.

4.2.x1 A chocolate bar is in the shape of a rectangle subdivided into squares as shown at right. How many breaks are needed to separate it completely into individual squares? Using the second principle of mathematical induction prove this: For any natural number $n$, a rectangular bar made up of $n$ individual pieces can be completely separated into individual squares in $n - 1$ breaks.

4.2.x2 The Fibonacci numbers are defined by the rules $F_1 = F_2 = 1$ and for every $n > 2$, $F_n$ is the sum of the two preceding Fibonacci numbers. That is, $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. The first several Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, ⋯ . Because each Fibonacci number depends on the two preceding numbers, induction arguments using the second principle of induction are often applied. Here are two examples of results that can be proved in this way.

a. For any natural number $n \geq 2$, $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$.

b. For any natural number $n$, $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n \cdot \sqrt{5}}$ .
[Note: For this proof, you have to verify the equation for both $n = 1$ and $n = 2$, as well as completing the induction step. Why?]

4.2.x3 A Fibonacci-like sequence is defined as follows: the first three terms are 1, 2, and 3, and every following term is the sum of the preceding *three* terms. This sequence begins 1, 2, 3, 6, 11, 20, 37, … . Let $a_n$ represent the $n^{th}$ term, so $a_1 = 1$, $a_2 = 2$, $a_5 = 11$, and so on. Use the second principle of induction to prove $a_n < 2^n$ for every natural number $n$. [How many base cases are necessary?]