FOM Lecture Notes. Day 1: Tuesday, 1/17/2017

Start of Term Info

1. Hand in start of semester student survey. If you haven't filled it in yet, please bring it to the next class
2. Webpage is at dankalman.net (take link for Foundations). I won't lecture on course policies in general
3. Regular Homework and Quiz Problems
   a. First assignment is due on our next meeting, 1/24. Check assignment sheet.
   b. Read about how homework counts, format
   c. I am checking for completion, not grading regular homework; solutions will be posted after the due date.
   d. Do homework and check answers to prepare for quiz problems.
   e. Quiz problems are take-home quizzes. Open notes and text, but no discussion with other students or searching for answers on the internet. Violating these rules will be considered to be cheating.
4. We're rushing though chapters 1-3 because these try to teach about proofs in the abstract and I would rather have students learn about proofs in the context of the topics we need to cover in depth. Students will have to read this material on their own, I will hit a few highlights and present supplementary material in class.
5. There are online lectures available at the website where you got the book.
   https://sites.google.com/site/mathematicalreasoning3ed/
   I will post answers to the progress check questions in the reading

Several Examples

1. In your notes, compute this integral: $\int_{\ln 1/3}^{\ln 2} f(x)\ dx$ where $f(x) = \frac{2e^{2x}}{(1-e^{2x})^2}$.

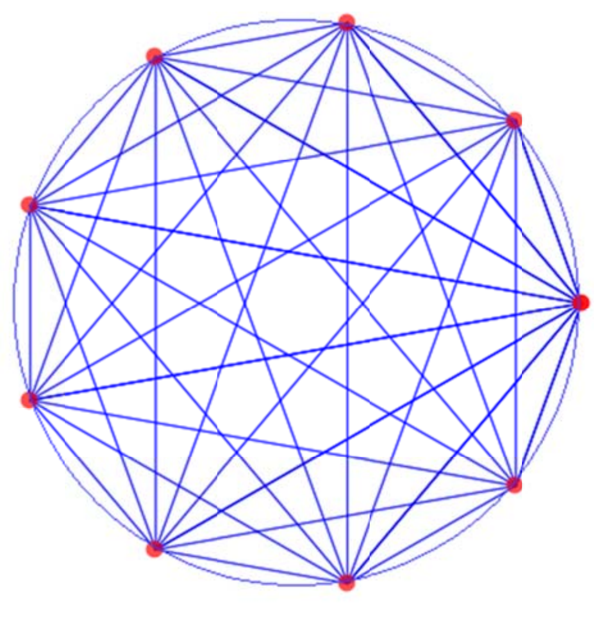
   [Hint: The antiderivative, $\frac{1}{1-e^{2x}} + C$, can be found with the substitution $u = e^{2x}$.]

   a. Did you find $\frac{1}{1-e^{2x}} \Big|_{\ln 1/3}^{\ln 2}$? After simplifying that equals -35/24 .
   b. That CANNOT be the correct value of the integral because $f(x)$ is never negative.
   c. Conclusion: The use of antiderivatives to find integrals doesn't always work. When DOES it?

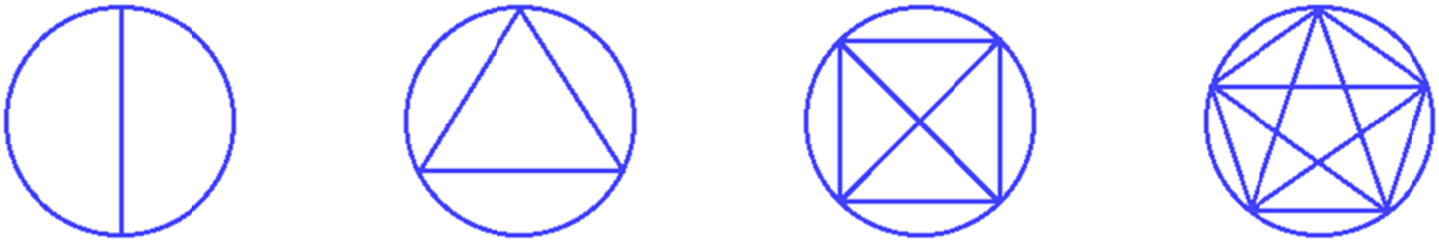
2. Use algebra to solve the equation $\ln x + 2\ln(1-x) = \ln 2$.
    a. By rules of logs, $\ln x + 2\ln(1-x) = \ln x + \ln(1-x)^2 = \ln[x(1-x)^2]$
    b. So we want to solve $\ln[x(1-x)^2] = \ln 2$.
    c. "Cancel" the ln on both sides: $x(1-x)^2 = 2$.
    d. Expand on the left to $x(x^2 - 2x + 1) = x^3 - 2x^2 + x$.
    e. So we want to solve $x^3 - 2x^2 + x - 2 = 0$.
    f. Factor the left by grouping:
       $x^3 - 2x^2 + x - 2 = x^2(x-2) + x - 2 = (x^2+1)(x-2)$.
    g. The first factor never equals 0. So there is exactly one solution $x = 2$.
    h. All of the preceding steps are correct.
    i. However, 2 is not a solution of the original equation.
    j. Conclusion: solving equations by algebra does not always produce solutions.

3. In your notes, sketch a graph for $x \geq 0$ for two functions, both of which are increasing and concave up.
    a. How many times do they cross?
    b. Can you draw such a graph with the curves crossing twice? Three times? More?
    c. Based on your work, would you agree with this "rule": If f and g are increasing and concave up functions for $x \geq 0$, then $f(x)$ can equal $g(x)$ for at most two values of x.
    d. OTOH … let $f(x) = x^2$. The first and second derivatives of f are both positive for positive x, so f is concave up and increasing for positive x.
    e. Let $g(x) = x^2 + \sin x$. Then $g'(x) = 2x + \cos x$, and $g''(x) = 2 - \sin x$. For all positive x , $g''(x) > 0$ because $\sin x$ can be at most 1. So g is concave up. Also, g' is increasing because its derivative is positive. And since $g'(0) = 1$, we can see that $g'(x)$ is always at least 1, and hence positive. This shows that g is increasing. Thus we have shown that g is increasing and concave up for $x \geq 0$.
    f. Now observe that $f(x) = g(x)$ for $x = 0, \pi, 2\pi, 3\pi, 4\pi$, etc. So the proposed rule above is wrong. In fact, two concave up increasing functions can cross an infinite number of times.
    g. Conclusion: Good guesses based on pictures can be wrong.

4. Mark 11 points equally spaced on a circle, and connect them with straight lines in all possible ways. How many regions inside the circle are formed?
    a. This is complicated to count, so let's systematically explore simpler cases
    b. If you mark just one point there will be no lines, so there will be just one region.



c. The figures below show the situation with 2, 3, 4, and 5 points



We can easily count the number of regions in each case, and enter the results in a table as follows:

Number or Points	1	2	3	4	5
Number of regions	1	2	4	8	16

Based on these results, a clear pattern emerges: the number of regions doubles with each additional point. In fact, for the cases in the table, the number of regions for *n* points is 2^{n-1}. This suggests that the number of regions for the original question with 11 points should be $2^{10} = 1024$.

d. How much confidence do you have in that prediction? Would you bet a dollar that it is right? $10? $100? $1000?

e. Actually, in the graphic for 11 points the number of regions doesn't *look* like it should be over 1000. And in fact, if you test the pattern with just one more point you find that there are only 30, not 32 regions. One easy way to count the regions in this case is to observe the six-fold symmetry. The pattern of green regions repeats six times around the central point. With 5 regions in each of these six patterns, there will be 30, not 32 regions. In fact, for any even *n* a similar symmetry will occur, with *n* identical wedges arranged around the central point. This shows that the number of regions must be a evenly divisible by *n*. Unless *n* is a power of 2, the number of regions cannot be 2^{n-1}.

f. Conclusion: Some patterns peter out!

What is the point of all these examples?

1. In math, we want to have certainty about our methods and conclusions. The examples show several ways that certainty has failed us. But in fact, proper mathematical analysis avoids these problems by insisting on extremely precise statements of mathematical knowledge, and rigorous logical proofs that these statements are correct. A major focus of this course is creating, using, and critiquing these kinds of statements and proofs. Now, more than in your prior courses, you need to pay attention to how we prove things in mathematics.

2. Mathematical knowledge usually concerns general rules – results that hold for a large number (probably infinitely many) instances. Theorems begin with statements such as

For any real number x, or *For any continuous function f* (x), or *For any triangle.* This is a way of specifying the instances for which the result holds. Our problem with Example 1 is we did not pay attention to the part of the fundamental theorem of calculus that says when the theorem can be applied. One version of the theorem says this: Suppose that $f(x)$ and $F(x)$ are functions and that $F'(x) = f(x)$ for every x in $[a,b]$. Then

$$\int_a^b f(x)dx = F(b) - F(a).$$

For the integral we considered, the interval is [ln 1/3 , ln 2], and that includes 0 because ln 1/3 < 0. However, our function $F(x) = \frac{1}{1-e^{2x}}$ is not defined at $x = 0$, so we cannot say that $F'(x) = f(x)$ for every x in $[a,b]$. This shows that the FTOC does not apply for the stated integral.

3. Mathematical proofs always include assumptions and conclusions. In its simplest form, the proof begins by assuming one or more things, and then derives several logical consequences of these assumptions, eventually reaching the desired conclusion. For the FTOC, the assumptions are that $f(x)$ and $F(x)$ are functions and that $F'(x) = f(x)$ for every x in $[a,b]$.

4. Logical consequences can be one-directional. If you assume that $x = -3$, then you can logically conclude that $x^2 = 9$. But the reverse is not true. If you know that $x^2 = 9$ you are not justified in inferring that $x = -3$. It might be true, or it might be that $x = 3$. So in a proof, if you have shown that $x^2 = 9$, you cannot next write down *Therefore* $x = -3$.

5. Let us look at the logic behind our algebraic steps in Example 2. There is an unspoken assumption at the beginning of the process: *Suppose x is a solution to the equation.* This says that the symbol x represents a particular (but as yet unknown) solution to the equation. Every step that follows is another true statement about that same x. So in outline form, here is the logic of the algebra:

   If x is a solution
   then $\ln x + 2\ln(1-x) = \ln 2$,
   then $\ln[x(1-x)^2] = \ln 2$,
   ⋮
   then $(x^2+1)(x-2) = 0$,
   then $x = 2$.

   So the nature of our conclusion is this: If x is a solution, then $x = 2$. But this does not actually say that 2 *IS* a solution. For that we would need the opposite logic, namely, *If x = 2 then x is a solution.* It turns out that some of the inferences in the algebraic steps are not reversible, and in fact, 2 is *not* a solution. You can verify this by trying to substitute $x = 2$ in the original equation.

   In general, the sort of algebraic procedure used in this example always leads to a conclusion of the form *If x is a solution, then x is …* . This can be expressed using the language of sets. For our example, what we showed is this:

The set of solutions is contained in {2}.

And in general, starting with an equation and then going through a series of rearrangements generally results in similar conclusion: *The set of solutions to the original equation is a subset of some specified set.* Ideally this will be a finite set, but that does not always occur.

If you are careful to check that every step of your algebra is logically reversible, then you can also reverse the conclusion. But even when that is not possible, the algebra serves to narrow down the range of possibilities. For our example, the algebra shows that there is only one possible solution, namely $x = 2$. When we find that this is NOT a solution we also find that there are NO solutions.

6. The third example shows that we cannot just jump to conclusions on the basis of what seems reasonable – not if we want to be certain. That is why we insist on specific logical proofs. And as you will find when you write your own proofs, you will be expected to have a specific justification for every statement you make.

7. The fourth example shows that a pattern of results cannot be accepted as proving a general rule. We know that the specific results we have found are true. We do not know that the pattern applies to cases we have not tested, unless we have a proof that shows this logically. Here is an example. If you add two odd numbers, you will observe that the result is always even. After many examples, you might propose this rule: If n and m are odd numbers, then $n+m$ is even. Notice that the first clause is the assumption. It says that the rule under consideration is supposed to be valid for any two odd numbers, and thus specifies which cases the rule applies to. Here is a proof of the statement:

   > Let n and m be odd numbers. Then $n - 1$ is even and $m + 1$ is even, because the even and odd integers alternate. Adding we see that $n - 1 + m + 1$ is the sum of two even numbers, and therefore is even. But $n - 1 + m + 1 = n+m$. Thus we have shown that $n+m$ is even.

   Because we want the result to apply for any two odd integers, we have to use variables. There are too many odd integers to consider them one at a time. The first line of the proof introduces the variables n and m and says, in essence, I don't know exactly what integers odd integers n and m are, but they must possess any property that is common to all odd integers. Each succeeding statement declares some additional bit of information about n and m is true. And finally we find that the desired conclusion is true.

Additional comments about proofs, time permitting

1. The first time a variable is mentioned in a proof, you are required to somehow specify the set of possible replacements. In the proof above, saying "Let n and m be odd numbers." means that any following statements I make about m and n must be valid for any odd integers.

2. Another way to say the same thing would be: "For all odd integers m and n …" But the flow of logic is conversationally easier to follow if you think of m and n as being specific quantities that will not change in the context of the argument.

3. Sometimes we introduce a variable to represent a new value that has special properties in relation to previously introduced variables. Here is an example. "Let n be a positive integer. We know that n must have a prime divisor p…" That is equivalent to saying that the permitted replacements for p are all the elements in the set of prime divisors of n. If you are thinking of n as a particular (though not known) positive integer, then the set of prime divisors of that integer is implicitly defined.

4. One way to test the validity of your logic is to choose specific permitted values for each variable as it is introduced, and then see if the logic remains correct. For the example above, after the line "Let n and m be odd numbers." you could say, ok, I will take n to be 37 and n to be -19. Now continue the proof with those values of n and m.

   "Then $n - 1$ (36) is even and $m + 1$ (-18) is even, because the even and odd integers alternate. Adding we see that $n - 1 + m + 1$ (36 + -18) is the sum of two even numbers, and therefore is even. But $n - 1 + m + 1$ (36 + -18) = n+m (37 + -19). Thus we have shown that n+m is even.

   If you test a proof in this way, and reach a statement that is false with your specific variable values, that indicates a logic error.

5. To emphasize the directionality of logical inference, we sometimes use a double arrow ⇒ as a short hand for "*and therefore*." In our algebra process, we can write out the steps this way:

   We assume x is a solution
   $\Rightarrow \ln x + 2\ln(1-x) = \ln 2,$
   $\Rightarrow \ln[x(1-x)^2] = \ln 2,$
   $\vdots$
   $\Rightarrow (x^2+1)(x-2) = 0,$
   $\Rightarrow x = 2.$

   This can be summarized as (x is a solution) ⇒ (x = 2). This is not done in formal mathematical writing (in books and journal articles) except in equations written in symbolic logic. But it can help you organize your thoughts.

End of Day