

Day 13: Tuesday, 2/28/2017

Hand back exams, explain exam solutions group project, have students form groups.

Take questions if any about any prior material.

Lecture Topic 1. Functions (6.1)

1. Overview

- a. Familiar idea of functions from calc and precalc
- b. More general definition for higher math – function from a set A to a set B .
- c. Examples: the Mother Function; the Grandmother non-Function; The factorial function; The birthday function.
- d. One metaphor: input output process – each input is an element of A , each output an element of B , every element of A is a valid input, and each such input produces only one output. (This is the vertical line test in calculus and precalc).
- e. Another metaphor: mapping from A to B : each element of A gets transported to or associated with an element of B . This can be represented with a mapping diagram or arrow diagram. Representative arrow diagrams for the examples above.

2. Book's definition

- a. A function from set A to set B is a rule that associates with each element of A exactly one element of B . This is also called a mapping from A to B .
- b. Note: this definition requires that there is a b for every a in A . It does not make any such requirement for elements of the co-domain. If b is in B , there may or may not be a corresponding a . That is, the function maps every a to something in B , but not every b is necessarily the destination of some a .
- c. Notation: $f: A \rightarrow B$. If $a \in A$, there is a unique element of B associated with a by the function f . This is usually denoted by $f(a)$.
- d. Terminology: If $f: A \rightarrow B$ the set A is called the domain of f and the set B is called the co-domain of f .
- e. Sometimes we write notation like $y = f(x)$ where x can be any element of the domain and y is then the corresponding element of the co-domain. In this context, x is said to be the independent variable because we can choose its value freely; y is the dependent variable because its choice depends on the value of x .
- f. For a particular a in A , we know that $f(a)$ is some element b in B , and we write $f(a) = b$. In this context, we say that b is **the** image of a and a is **a** pre-image of b . Note that an a can have only one image, but a particular b may have several preimages, or no preimage at all. Example: a constant function $f: \mathbb{N} \rightarrow \mathbb{N}$ with $f(n) = 2$ for every n in \mathbb{N} . Each n has just one image (namely 2); and every n is a preimage of 2; but there is no preimage of 3.

- g. In the extra exercises, the concept of image and preimage is extended from single elements to arbitrary subsets of the domain and codomain. If C is a subset of the domain A , the image of C is the set of all the images of C 's elements. We denote this by

$$f(C) = \{f(a) \mid a \in C\}.$$

Similarly, if D is a subset of the codomain B , then the preimage (also called the *inverse image*) of D is the set consisting of all the elements in the domain that map to D . This is denoted by $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$, and may be the empty set.

- h. The set $f(A)$ is the image of A , also called the range of A . This is a subset of B , and can in many cases be a proper subset – that is, is a subset of B but is not all of B . For example, with $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^2$, the range is $\{y \in \mathbb{R} \mid y \geq 0\}$. This is a proper subset of the codomain \mathbb{R} . The codomain consists of all the reals, positive, negative, and zero; while the range only consists of the non-negative reals.

3. Alternate definition

- The word *rule* in the book's definition is a problem – what exactly is a rule? Might different people have different ideas about whether something is or is not a rule?
- A more generally accepted definition in higher math defines the concept of function in terms of certain types of sets, thus avoiding the ambiguous word *rule*.
- A brief excerpt from another text is posted with today's lecture notes to develop this idea in detail.
- The basic idea is this: We consider a function from A to B to be a set of ordered pairs (a,b) – that is, a subset C of the Cartesian product $A \times B$ – with this property: For every a in A , there exists exactly one b in B such that (a,b) is an element of C . Restated: we are looking at a set $C \subseteq A \times B$ in which every a in A appears as the first element of one and only one of the ordered pairs in C . Such a set is called a function from A to B .
- In the book's definition, given a function $f: A \rightarrow B$ we can certainly define the set of ordered pairs $\{(a, f(a)) \mid a \in A\}$ and that will be a subset of $A \times B$. This is called the *Graph* of the function. But in the alternate formulation, this set of ordered pairs *IS* the function, so that the concept of a function is identified with its graph.

Lecture Topic 2. Modular Arithmetic

1. Overview

- This is a standard tool for working with integers.
- So far, I have said nothing about this in lecture and de-emphasized it in homework.
- Now we will return to this topic and use it frequently in examples.

2. Viewpoint 1: operations on integers

- Two integers are congruent modulo n if they leave the same remainder when divided by n
- Alternate version: a and b are congruent modulo n if their difference is divisible by n .

- c. In symbols: $a \equiv b \pmod{n}$ iff $n \mid b - a$.
- d. Example: Consider two integers representing hours of elapsed time from some predefined starting point. So 5 represents a time 5 hours after the starting point, and -3 represents the time 3 hours before the starting point. If n and m are two integers and are congruent modulo 12, then they represent the same time on a clock: the difference between them is a multiple of 12, so whatever the position of the clock hands is at time n , there will be a whole number of complete 12 hour cycles between n and m so the clock hands will look the same at time m as at n .
- e. Using congruences can sometimes allow us to work with a finite number of possibilities in place of the infinitely many integers. For example, suppose we wish to prove that there are no integer solutions to the equation $x^5 - 3x^2 - 8x = 3$. We can use a case argument as follows. Let x be any integer. Then there are two cases: either $x \equiv 0 \pmod{2}$ or $x \equiv 1 \pmod{2}$. This is because x is either even (and hence $x - 0$ divisible by 2) or $x - 1$ is even and hence $x - 1$ is divisible by 2. So consider case 1 -- suppose that $x \equiv 0 \pmod{2}$. Then $x^5 \equiv 0^5 \pmod{2}$ so $x^5 \equiv 0 \pmod{2}$. Similar reasoning shows that $x^2 \equiv 0 \pmod{2}$. That in turn implies that $3 \cdot x^2 \equiv 3 \cdot 0 = 0 \pmod{2}$. Continuing in this way shows that $x^5 - 3x^2 - 8x \equiv 0 - 0 - 0 = 0 \pmod{2}$. On the other hand $3 \not\equiv 0 \pmod{2}$. Thus, since $x^5 - 3x^2 - 8x \not\equiv 3 \pmod{2}$, $x^5 - 3x^2 - 8x$ cannot equal 3. This shows that in case 1, x cannot be a solution to the equation.

For case 2, we assume that $x \equiv 1 \pmod{2}$. In this case we derive, as before, that $x^5 - 3x^2 - 8x \equiv 1^5 - 3 \cdot 1^2 - 8 \cdot 1 = -10 \pmod{2}$. And since -10 is divisible by 2, it in turn is congruent to 0 mod 2. Thus, we again find that $x^5 - 3x^2 - 8x \equiv 0 \pmod{2}$, and again conclude that $x^5 - 3x^2 - 8x$ cannot equal 3. Therefore in case 2 x cannot be a solution to the equation. Thus we have shown that every integer x cannot be a solution to the equation.

3. Viewpoint 2: New number systems, related to but distinct from \mathbb{Z} .
- a. We define $\mathbb{Z}_n = \{0, 1, 2, \dots, n - 1\}$ and redefine addition and multiplication for this set.
- b. As a precursor, we define reduction mod n as follows. For any integer k , the remainder after dividing by n is one of the elements of \mathbb{Z}_n . Denote this as $\text{REM}(k \div n)$.
- c. We add elements in \mathbb{Z}_n according to the rule: $p +_n q = \text{REM}((p+q) \div n)$. That is, we add p and q using normal integer addition, and then compute the remainder of the total after dividing by n .
- d. We multiply elements in \mathbb{Z}_n according to the rule: $p \times_n q = \text{REM}((p \times q) \div n)$. That is, we multiply p and q using normal integer multiplication, and then compute the remainder of the total after dividing by n .
- e. In practice, just add or multiply in the usual way, and then subtract off the nearest smaller multiple of n .

- f. It can be shown that these operations obey all of the usual rules of arithmetic: commutative, associative, and distributive laws, etc. We usually just write $+$ and \times rather than $+_n$ and \times_n when we are working in the \mathbb{Z}_n number system.
4. There are many applications for these number systems. In particular, we often are interested in functions of the form $f: \mathbb{Z}_n \rightarrow \mathbb{Z}_n$, or $f: \mathbb{Z}_n \rightarrow \mathbb{Z}_m$ with $m \neq n$, or $f: \mathbb{Z}_n \rightarrow \mathbb{Z}$, or $f: \mathbb{Z} \rightarrow \mathbb{Z}_n$. Here is one example: define $f: \mathbb{Z}_6 \rightarrow \mathbb{Z}_6$ by the equation $f(x) = x^2 + 3$. We can compute $f(x)$ for each x in $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ to complete the following table:

x	0	1	2	3	4	5
$f(x)$	3	4	1	0	1	4

Lecture Topic 3. More Examples and properties of Functions (As many as time permits)

1. Equality of two functions

- For equality we have to have $f: A \rightarrow B$ and $g: A \rightarrow B$, and $f(a) = g(a)$ for all a in A .
- That is, we do not consider two functions to be identical unless they have the same domain and codomain.
- In terms of the alternate definition of function, f and g will each be a subset of $A \times B$ and they are equal functions if they are equal as sets.

2. Linear transformation from \mathbb{R}^n to \mathbb{R}^m .

- Defined in terms of a constant $m \times n$ matrix A .
- The function is defined by $f(\mathbf{x}) = A\mathbf{x}$, where \mathbf{x} is a column vector $[x_1 \ x_2 \ x_3 \ \dots \ x_n]^T$ and the result on the right side of the equality is given by matrix multiplication.
- This function satisfies the linearity conditions: $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for all vectors \mathbf{x} and \mathbf{y} ; and $f(c\mathbf{x}) = cf(\mathbf{x})$ for all vectors \mathbf{x} and all real scalars c .

3. The derivative as a function.

- Take the domain A to be the set of differentiable functions on the interval $[0,1]$
- Take the co-domain B to be the set of all real functions on $[0,1]$
- The mapping is depicted visually as $f \rightarrow f'$.
- If we represent this function by the letter D , then we can write $D: A \rightarrow B$ where, for every $f \in A$, $D(f) = f'$.
- A similar kind of mapping: Let $A = \{\text{all functions from } \mathbb{R} \text{ to } \mathbb{R}\}$ and define a mapping $T: A \rightarrow A$ as follows. For any f in A , let $T(f) = g$ where $g(x)$ is defined to equal $f(x+1)$ for every real x . In this example, if you think of the function f as being identical to its graph in the xy plane, then T has a graphical interpretation: it shifts the graph one unit to the left.

4. Sequences as Functions: a sequence is a function with domain equal to \mathbb{N} or some subset of \mathbb{Z} of the form $\{n, n+1, n+2, \dots\}$ where n is a fixed element of \mathbb{Z} . For example, we could

take the domain to be $\{-3, -2, -1, 0, 1, \dots\}$, and the sequence is then given by the terms $a_{-3}, a_{-2}, a_{-1}, a_0, \dots$. This is simply using subscript notation a_n in place of the more recognizable function notation $a(n)$.

5. Functions of 2 or more variables

- a. Notation: if we think of the domain as being made up of ordered pairs (x, y) , the literal extension of our function notation should be to write $f((x, y))$. But that is needlessly pedantic, so we make the notational convention of writing $f(x, y)$.
- b. Arithmetic operations are actually functions. For example, addition is the mapping $(x, y) \rightarrow x + y$.
- c. For functions from \mathbb{R}^2 to \mathbb{R} , the preimage of a point is often a curve. For example, consider the function $f(x, y) = x^2 - 3y$. We can compute $f(4, 2) = 10$. So the image of the point $(4, 2)$ is the number 10. On the other hand, the preimage of the number 10 is the set of all the points that f takes to 10. Using the definition of f , we see that (x, y) is such a point if and only if $x^2 - 3y = 10$. This is algebraically equivalent to $y = (x^2 - 10)/3$, and every point on that curve is a preimage of 10.

End of Day