Day 13: Tuesday, 2/28/2017

Hand back exams, explain exam solutions group project, have students form groups.

Take questions if any about any prior material.

Lecture Topic 1. Functions (6.1)

1. Overview
   a. Familiar idea of functions from calc and precalc
   b. More general definition for higher math – function from a set *A* to a set *B*.
   c. Examples: the Mother Function; the Grandmother non-Function; The factorial function; The birthday function.
   d. One metaphor: input output process – each input is an element of *A*, each output an element of *B*, every element of *A* is a valid input, and each such input produces only one output. (This is the vertical line test in calculus and precalc).
   e. Another metaphor: mapping from *A* to *B*: each element of *A* gets transported to or associated with an element of *B*. This can be represented with a mapping diagram or arrow diagram. Representative arrow diagrams for the examples above.
2. Book's definition
   a. A function from set *A* to set *B* is a rule that associates with each element of *A* exactly one element of *B*. This is also called a mapping from *A* to *B*.
   b. Note: this definition requires that there is a *b* for every *a* in *A*. It does not make any such requirement for elements of the co-domain. If *b* is in *B*, there may or may not be a corresponding *a*. That is, the function maps every *a* to something in *B*, but not every *b* is necessarily the destination of some *a*.
   c. Notation: $f : A \to B$. If $a \in A$, there is a unique element of *B* associated with *a* by the function *f* . This is usually denoted by $f(a)$.
   d. Terminology: If $f : A \to B$ the set *A* is called the domain of *f* and the set *B* is called the co-domain of *f* .
   e. Sometimes we write notation like $y = f(x)$ where *x* can be any element of the domain and *y* is then the corresponding element of the co-domain. In this context, *x* is said to be the independent variable because we can choose its value freely; *y* is the dependent variable because its choice depends on the value of *x*.
   f. For a particular *a* in *A*, we know that $f(a)$ is some element *b* in *B*, and we write $f(a) = b$. In this context, we say that *b* is **<u>the</u>** image of *a* and *a* is **<u>a</u>** pre-image of *b*. Note that an *a* can have only one image, but a particular *b* may have several preimages, or no preimage at all. Example: a constant function $f : \mathbb{N} \to \mathbb{N}$ with $f(n) = 2$ for every *n* in $\mathbb{N}$. Each *n* has just one image (namely 2); and every *n* is a preimage of 2; but there is no preimage of 3.

g. In the extra exercises, the concept of image and preimage is extended from single elements to arbitrary subsets of the domain and codomain. If C is a subset of the domain A, the image of C is the set of all the images of C's elements. We denote this by
$$f(C) = \{f(a) \mid a \in c\}.$$
Similarly, if D is a subset of the codomain B, then the preimage (also called the *inverse* image) of D is the set consisting of all the elements in the domain that map to D. This is denoted by $f^{-1}(D) = \{a \in A \mid f(a) \in D\}$, and may be the empty set.

h. The set $f(A)$ is the image of A, also called the range of A. This is a subset of B, and can in many cases is a proper subset – that is, is a subset of B but is not all of B. For example, with $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$, the range is $\{y \in \mathbb{R} \mid y \geq 0\}$. This is a proper subset of the codomain $\mathbb{R}$. The codomain consists of all the reals, positive, negative, and zero; while the range only consists of the non-negative reals.

3. Alternate definition

a. The word *rule* in the books definition is a problem – what exactly is a rule? Might different people have different ideas about whether something is or is not a rule?

b. A more generally accepted definition in higher math defines the concept of function in terms of certain types of sets, thus avoiding the ambiguous word *rule*.

c. A brief excerpt from another text is posted with today's lecture notes to develop this idea in detail.

d. The basic idea is this: We consider a function from A to B to be a set of ordered pairs (a,b) – that is, a subset C of the Cartesian product $A \times B$ – with this property: For every a in A, there exists exactly one b in B such that (a,b) is an element of C. Restated: we are looking at a set $C \subseteq A \times B$ in which every a in A appears as the first element of one and only one of the ordered pairs in C. Such a set is called a function from A to B.

e. In the book's definition, given a function $f: A \to B$ we can certainly define the set of ordered pairs $\{(a, f(a)) \mid a \in A\}$ and that will be a subset of $A \times B$. This is called the *Graph* of the function. But in the alternate formulation, this set of ordered pairs *IS* the function, so that the concept of a function is identified with its graph.

Lecture Topic 2. Modular Arithmetic

1. Overview

a. This is a standard tool for working with integers.

b. So far, I have said nothing about this in lecture and de-emphasized it in homework.

c. Now we will return to this topic and use it frequently in examples.

2. Viewpoint 1: operations on integers

a. Two integers are congruent modulo n if they leave the same remainder when divided by n

b. Alternate version: a and b are congruent modulo n if their difference is divisible by n.

c. In symbols: $a \equiv b$ (mod n) iff $n \mid b - a$.

d. Example: Consider two integers representing hours of elapsed time from some predefined starting point. So 5 represents a time 5 hours after the starting point, and -3 represents the time 3 hours before the starting point. If n and m are two integers and are congruent modulo 12, then they represent the same time on a clock: the difference between them is a multiple of 12, so whatever the position of the clock hands is at time n, there will be a whole number of complete 12 hour cycles between n and m so the clock hands will look the same at time m as at n.

e. Using congruences can sometimes allow us to work with a finite number of possibilities in place of the infinitely many integers. For example, suppose we wish to prove that there are no integer solutions to the equation $x^5 - 3x^2 - 8x = 3$. We can use a case argument as follows. Let x be any integer. Then there are two cases: either $x \equiv 0$ (mod 2) or $x \equiv 1$ (mod 2). This is because x is either even (and hence $x - 0$ divisible by 2) or $x - 1$ is even and hence $x - 1$ is divisible by 2. So consider case 1 -- suppose that $x \equiv 0$ (mod 2). Then $x^5 \equiv 0^5$ (mod 2) so $x^5 \equiv 0$ (mod 2). Similar reasoning shows that $x^2 \equiv 0$ (mod 2). That in turn implies that $3 \cdot x^2 \equiv 3 \cdot 0 = 0$ (mod 2). Continuing in this way shows that $x^5 - 3x^2 - 8x \equiv 0 - 0 - 0 = 0$ (mod 2). On the other hand
$3 \not\equiv 0$ (mod 2). Thus, since $x^5 - 3x^2 - 8x \not\equiv 3$ (mod 2), $x^5 - 3x^2 - 8x$ cannot equal 3. This shows that in case 1, x cannot be a solution to the equation.

For case 2, we assume that $x \equiv 1$ (mod 2). In this case we derive, as before, that $x^5 - 3x^2 - 8x \equiv 1^5 - 3 \cdot 1^2 - 8 \cdot 1 = -10$ (mod 2). And since -10 is divisible by 2, it in turn is congruent to 0 mod 2. Thus, we again find that $x^5 - 3x^2 - 8x \equiv 0$ (mod 2), and again conclude that $x^5 - 3x^2 - 8x$ cannot equal 3. Therefore in case 2 x cannot be a solution to the equation. Thus we have shown that every integer x cannot be a solution to the equation.

3. Viewpoint 2: New number systems, related to but distinct from $\mathbb{Z}$.

a. We define $\mathbb{Z}_n = \{0, 1, 2, \dots , n - 1\}$ and redefine addition and multiplication for this set.

b. As a precursor, we define reduction mod n as follows. For any integer k, the remainder after dividing by n is one of the elements of $\mathbb{Z}_n$. Denote this as REM($k \div n$).

c. We add elements in $\mathbb{Z}_n$ according to the rule: $p +_n q$ = REM((p+q) $\div$ n). That is, we add p and q using normal integer addition, and then compute the remainder of the total after dividing by n.

d. We multiply elements in $\mathbb{Z}_n$ according to the rule: $p \times_n q$ = REM((p×q) $\div$ n). That is, we multiply p and q using normal integer multiplication, and then compute the remainder of the total after dividing by n.

e. In practice, just add or multiply in the usual way, and then subtract off the nearest smaller multiple of n.

f. It can be shown that these operations obey all of the usual rules of arithmetic: commutative, associative, and distributive laws, etc. We usually just write + and × rather than $+_n$ and $\times_n$ when we are working in the $\mathbb{Z}_n$ number system.

4. There are many applications for these number systems. In particular, we often are interested in functions of the form $f : \mathbb{Z}_n \to \mathbb{Z}_n$, or $f : \mathbb{Z}_n \to \mathbb{Z}_m$ with $m \neq n$, or $f : \mathbb{Z}_n \to \mathbb{Z}$, or $f : \mathbb{Z} \to \mathbb{Z}_n$. Here is one example: define $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ by the equation $f(x) = x^2 + 3$. We can compute $f(x)$ for each x in $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ to complete the following table:

x	0	1	2	3	4	5
$f(x)$	3	4	1	0	1	4

Lecture Topic 3. More Examples and properties of Functions (As many as time permits)

1. Equality of two functions
   a. For equality we have to have $f : A \to B$ and $g : A \to B$, and $f(a) = g(a)$ for all a in A.
   b. That is, we do not consider two functions to be identical unless they have the same domain and codomain.
   c. In terms of the alternate definition of function, f and g will each be a subset of $A \times B$ and they are equal functions if they are equal sets.

2. Linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.
   a. Defined in terms of a constant $m \times n$ matrix A.
   b. The function is defined by $f(\mathbf{x}) = A\mathbf{x}$, where $\mathbf{x}$ is a column vector $[x_1 \ \ x_2 \ \ x_3 \ \cdots \ x_n]^T$ and the result on the right side of the equality is given by matrix multiplication.
   c. This function satisfies the linearity conditions: $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$ for all vectors $\mathbf{x}$ and $\mathbf{y}$; and $f(c\mathbf{x}) = c\,f(\mathbf{x})$ for all vectors $\mathbf{x}$ and all real scalars c.

3. The derivative as a function.
   a. Take the domain A to be the set of differentiable functions on the interval [0,1]
   b. Take the co-domain B to be the set of all real functions on [0,1]
   c. The mapping is depicted visually as $f \to f'$.
   d. If we represent this function by the letter D, then we can write $D : A \to B$ where, for every $f \in A$, $D(f) = f'$.
   e. A similar kind of mapping: Let $A = \{$all functions from $\mathbb{R}$ to $\mathbb{R}\}$ and define a mapping $T : A \to A$ as follows. For any f in A, let $T(f) = g$ where $g(x)$ is defined to equal $f(x+1)$ for every real x. In this example, if you think of the function f as being identical to its graph in the xy plane, then T has a graphical interpretation: it shifts the graph one unit to the left.

4. Sequences as Functions: a sequence is a function with domain equal to $\mathbb{N}$ or some subset of $\mathbb{Z}$ of the form $\{n, n+1, n+2, \cdots\}$ where n is a fixed element of $\mathbb{Z}$. For example, we could

take the domain to be {-3, -2, -1, 0, 1, ⋯ }, and the sequence is then given by the terms a_{-3}, a_{-2}, a_{-1}, a_0, ⋯ . This is simply using subscript notation a_n in place of the more recognizable function notation $a(n)$.

5. Functions of 2 or more variables
   a. Notation: if we think of the domain as being made up or ordered pairs (x, y), the literal extension of our function notation should be to write $f((x, y))$. But that is needlessly pedantic, so we make the notational convention of writing $f(x, y)$.

   b. Arithmetic operations are actually functions. For example, addition is the mapping $(x, y) \rightarrow x + y$.

   c. For functions from $\mathbb{R}^2$ to $\mathbb{R}$, the preimage of a point is often a curve. For example, consider the function $f(x, y) = x^2 - 3y$. We can compute $f(4,2) = 10$. So the image of the point (4,2) is the number 10. On the other hand, the preimage of the number 10 is the set of all the points that f takes to 10. Using the definition of f, we see that (x, y) is such a point if and only if $x^2 - 3y = 10$. This is algebraically equivalent to $y = (x^2 - 10)/3$, and every point on that curve is a preimage of 10.

End of Day