Day 16: Friday, 3/10/2017

Collect 6.3 homework if students are ready. Collect late quiz problems from prior class. Take questions. Announce: If I cover all or most of 6.4 today there will be a regular homework assignment from that section due after spring break. In any case, the quiz problem from 6.3 will be due after spring break as is already indicated on the assignment sheet.

Continue prior class's lecture

1. Surjections
   a. Review definition of surjection
   b. Observation, if $f : A \to B$ is not surjection, we can redefine the codomain to be the image of *A* (ie the range) and that new function will be surjective. When the range is a simple set and we know exactly what it is, this is practical. OTOH, if the range is complicated, this may not be very useful. Example: $f : \mathbb{R} \to \mathbb{R}$ defined as follows: for each real *x* express it as a decimal expansion (that doesn't terminate in an infinite string of 9s) and change every 5 to a 6. What is the range? Summary: the property of being surjective is most useful when the image is some well understood and/or familiar set.
   c. To prove that *f* is *not* a surjection, exhibit one element *b* of *B* that has no preimage in *A*.
   d. Example: $f : \mathbb{Z}_6 \to \mathbb{Z}_6$ defined by the equation $f(x) = x^2 + 3$ is not surjective because $f(a)$ never equals 5.
   e. To prove that *f* IS a surjection, follow this outline: Assume that *b* is an arbitrary element of *B* and demonstrate that there exists an *a* in *A* for which $f(a) = b$.
   f. Example: Let $f : [0,1] \to [0,1]\times[0,1]$ defined as follows: First, $f(1)$ is defined explicitly to be (1,1). Next. each *x* in [0,1) has a decimal expansion of the form $x = 0.d_1d_2d_3d_4 \cdots$ which does not end in an infinite string of 9's. (This condition is necessary so that each *x* has just *one* decimal expansion. For example 1/2 can be expressed as either 0.5 or as $0.49999 \cdots$, but we exclude the one that ends in an infinite string of 9's.) Define $f(0.d_1d_2d_3d_4 \cdots) = (0.d_1d_3d_5d_7 \cdots, 0.d_2d_4d_6d_8 \cdots)$. For example, since $0 = 0.0000\cdots$, we see that $f(0) = (0,0)$. Similarly, $f(.192959393939393939 \cdots)$ equals $(0.125333\cdots, 0.999999\cdots) = (0.125333\cdots, 1) \in [0,1]\times[0,1]$.
      Now claim that this function is a surjection. To prove this, we let (*y*,*z*) be an arbitrary element of $[0.1]\times[0.1]$ and show that $(y,z) = f(x)$ for some *x* in [0,1]. To do this, we consider four cases: (1) both *y* and *z* in [0,1); (2) *y* in [0,1) and $z = 1$; (3) *z* in [0,1) and $y = 1$; and (4) $y = z = 1$.

      In case 1, both *y* and *z* have decimal expansions so we can write $y = 0.y_1y_2y_3y_4 \cdots$ and $z = 0.z_1z_2z_3z_4 \cdots$ where neither expansion ends in an infinite string of 9's. In this case observe that $f(0.y_1z_1y_2z_2y_3z_3y_4z_4 \cdots) = (y,z)$

      In case 2, write $y = 0.y_1y_2y_3y_4 \cdots$ and observe that
      $$f(0.y_19y_29y_39y_49 \cdots) = (y, 0.999\cdots) = (y,1) = (y,z).$$
      Moreover, note that $0.y_19y_29y_39y_49 \cdots$ does not end in an infinite string of 9's, because $y = 0.y_1y_2y_3y_4 \cdots$ does not end in an infinite string of 9's.

In case 3, write $z = 0.z_1z_2z_3z_4 \cdots$ and observe that

$$f\,(0.9z_19z_29z_39z_4 \cdots) = (0.999 \cdots, z) = (1,z) = (y,z).$$

As in case 2, we also note that $0.9z_19z_29z_39z_4 \cdots$ does not end in an infinite string of 9's, because $z = 0.z_1z_2z_3z_4 \cdots$ does not end in an infinite string of 9's.

Finally, in case 4, we know by definition that $f\,(1) = (1,1) = (y,z)$.

Thus, in each case we have exhibited an x for which $f\,(x) = (y,z)$, showing that f is surjective.

Comment: This function can actually be proven to be continuous. So we have a continuous function from [0,1] onto the rectangle $[0,1]^2$. It is not one-one.

2. Book example: Linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Example shows how to verify that this function is a bijection. Won't discuss this in detail today.

3. Injections and Surjections for finite domains
   a. If A has n elements, then at most n $f\,(a)$ values are possible
   b. If A is an injection, then the $f\,(a)$ values are all distinct so $f\,(A)$ will have exactly n values. In this case B must have n or more values, and f is a surjection iff B has exactly n values.
   c. Using similar reasoning, if f is a surjection and B has n elements then A has to have n or more elements, and f is an injection iff A has exactly n values.
   d. In fact, any two of the following statements implies the third:
      i. f is an injection
      ii. f is a surjection
      iii. A and B have the same number of elements
   e. A bijection is also called a one-to-one correspondence, and represents an exact pairing of the sets A and B. For finite sets finding a bijection is one way to see that two sets have the same number of elements
   f. Revisit proof that if A has n elements then the power set of A has 2^n elements. We argued by induction, and in the induction step we split the subsets of $\{x_1, x_2, x_2, \cdots, x_{n+1}\}$ into two types: those that include x_{n+1} and those that do not. The sets that do not involve x_{n+1} are simply the elements of the power set of $\{x_1, x_2, x_2, \cdots, x_n\}$ and by the induction hypothesis there are 2^n of those. We want to show that there are also 2^n sets of the other type , accounting for 2^{n+1} overall. For this purpose we define a mapping from the type 1 sets to the type 2 sets and show that this is a bijection. The mapping is simple to define: For every set S of the first type, we define $f(S) = S \cup \{x_{n+1}\}$. To complete the argument we have to show that this mapping is a bijection.
   g. For infinite sets, one-to-one correspondences are the key concept in comparing sizes of sets. Curious example the set of integers $\mathbb{Z}$ and the set of even integers $2\mathbb{Z}$ are linked by

the one to one correspondence $f(n) = 2n$. So these two sets are the same *size* even though one is a proper subset of the other. OTOH, there is no one-to-one correspondence between $\mathbb{Z}$ and $\mathbb{R}$. We will see this at the end of the course.

New Topic Composition of Functions

1. Definition of composition of Two Functions
   a. If $f : A \to B$ and $g : B \to C$ then we can define $g \circ f : A \to C$ by the equation
      $g \circ f(a) = g(f(a))$
   b. Note reversal of order
   c. Arrow diagram explanation
   d. Definition also makes sense if $g : f(A) \to C$, though our book does not include a definition of this.

2. Theorem 6.20. Let A, B, and C be nonempty sets and f and g functions with $f : A \to B$ and $g : B \to C$.
   a. Part 1: If f and g are both injections, so is $g \circ f : A \to C$.
   b. Part 2: If f and g are both surjections, so is $g \circ f : A \to C$.
      Proof:
      i. Outline: start with "Let c be any element of C", in the middle somewhere define an element of a of A, and end with "Therefore $g \circ f(a) = c$."
      ii. So, c is in C, and g is a surjection, so $c = g(b)$ for some b in B. And now, since f is a surjection, $b = f(a)$ for some a in A. Therefore, $g \circ f(a) = g(f(a)) = g(b) = c$.
   c. Part 3: If f and g are both bijections, so is $g \circ f : A \to C$.
   d. These are not iff statements. Example, if $A = B = C = \mathbb{R}$, $f(x) = x^2$, and $g(x) = x \sin x$, then $g \circ f$ is surjective but f is not. Similarly, if $A = \{1, 2, 3\}$, $B = \{101, 102\}$, and $C = \{0\}$, and f and g are both constant functions, with $f(a) = 101$ for all a and $g(b) = 0$ for all b. (Make an arrow diagram for this example). On the other hand, if $g \circ f$ is surjective then g must definitely be surjective. (Why?) Theorem 6.21 relates to these examples, and the idea that the statements in Theorem 6.20 are not iff statements.

3. Special example: Let n be a given natural number. Define $A = B = C = \{1, 2, 3, \cdots, n\}$. If f and g are any two bijections from A to A then so is their composition. Consider the set of all possible bijections from A to A. These are called permutations of A, and each permutation can be considered as a way to mixup or reorder the elements of A. The set of permutations is denoted S_n and has many interesting properties and applications. Note that the identity function defined by $id(x) = x$ for all x is an element of S_n, and that for any other permutation f, $id \circ f = f$ and $f \circ id = f$. Also, S_n is closed under the operation of composition. This operation is associative, but not commutative. It is also noteworthy that the number of elements in S_n is 2^n. All of these statements can be proved using methods we have discussed in this course. The last is provable by induction.

End of Day