Day 17: Tuesday, 3/21/2017

Collect 6.3 homework. Take questions.

Revisit the proof of this proposition: if A has n elements then the power set of A has 2^n elements.

1. We argued by induction, and in the induction step we split the subsets of $\{x_1, x_2, x_2, \cdots, x_{n+1}\}$ into two types: those that include x_{n+1} and those that do not. The sets that do not involve x_{n+1} are simply the elements of the power set of $\{x_1, x_2, x_2, \cdots, x_n\}$ and by the induction hypothesis there are 2^n of those. We want to show that there are also 2^n sets of the other type, accounting for 2^{n+1} overall. For this purpose we define a mapping from the type 1 sets to the type 2 sets and show that this is a bijection. The mapping is simple to define: For every set S of the first type, we define $f(S) = S \cup \{x_{n+1}\}$. To complete the argument we have to show that this mapping is a bijection.

2. There is an alternate proof that uses bijections in a different way. We compare the power set of A with the set $\mathcal{F}$ of functions $f : \{\in, \notin\} \to A$.
   a. There are 2^n such functions, because for each $a \in A$, we can either define $f(a)$ to be $\in$ or $\notin$. Thus we have two different ways to determine n independent choices. This shows that there 2^n elements of $\mathcal{F}$. (This is like the possible outcomes from flipping a coin n times.)
   b. ★ Digression: If C has m elements and D has n elements, there are n^m different functions from C to D. If C and D each have m elements, there are $m!$ *bijections* from C to D.
   c. Each element of $\mathcal{F}$ can be represented by a table of the form

x	a_1	a_2	$\cdots$	a_n
$f(x)$	$f(a_1)$	$f(a_2)$	$\cdots$	$f(a_n)$

   where $a_1, a_2, \cdots, a_n$ are the distinct elements of A and each $f(x)$ is either $\in$ or $\notin$.
   d. For example, if $n = 5$, one such function in $\mathcal{F}$ is given by the table

x	a_1	a_2	a_3	a_4	a_5
$f(x)$	$\notin$	$\in$	$\in$	$\notin$	$\in$

   e. Notice that this table defines a subset of A in a natural way. The table in our example defines the subset $\{a_2, a_3, a_5\}$.
   f. More generally, for every $f \in \mathcal{F}$, there is a subset S_f of A defined by the rule

$$(\forall\, a \in A)\,(a \in S_f \text{ iff } f(a) = \in.$$

g. This establishes a bijection between $\mathcal{F}$ and the power set $\wp(A)$. Since we have already argued that $|\mathcal{F}| = 2^n$, the bijection shows that $|\wp(A)| = 2^n$, too.

New Topic Composition of Functions

1. Definition of composition of Two Functions
   a. If $f : A \rightarrow B$ and $g : B \rightarrow C$ then we can define $g \circ f : A \rightarrow C$ by the equation
      $g \circ f(a) = g(f(a))$
   b. Note reversal of order
   c. Arrow diagram explanation
   d. Definition also makes sense if $g : f(A) \rightarrow C$, though our book does not include a definition of this.
2. Theorem 6.20. Let A, B, and C be nonempty sets and f and g functions with $f : A \rightarrow B$ and $g : B \rightarrow C$.
   a. Part 1: If f and g are both injections, so is $g \circ f : A \rightarrow C$.
   b. Part 2: If f and g are both surjections, so is $g \circ f : A \rightarrow C$.
      Proof:
      i. Outline: start with "Let c be any element of C", in the middle somewhere define an element of a of A, and end with "Therefore $g \circ f(a) = c$."
      ii. So, c is in C, and g is a surjection, so $c = g(b)$ for some b in B. And now, since f is a surjection, $b = f(a)$ for some a in A. Therefore, $g \circ f(a) = g(f(a)) = g(b) = c$.
   c. Part 3: If f and g are both bijections, so is $g \circ f : A \rightarrow C$.
   d. These are not iff statements. Example, if $A = B = C = \mathbb{R}$, $f(x) = x^2$, and $g(x) = x \sin x$, then $g \circ f$ is surjective but f is not. Similarly, if $A = \{1, 2, 3\}$, $B = \{101, 102\}$, and $C = \{0\}$, and f and g are both constant functions, with $f(a) = 101$ for all a and $g(b) = 0$ for all b. (Make an arrow diagram for this example). On the other hand, if $g \circ f$ is surjective then g must definitely be surjective. (Why?) Theorem 6.21 relates to these examples, and the idea that the statements in Theorem 6.20 are not iff statements.

3. Special example: Let n be a given natural number. Define $A = B = C = \{1, 2, 3, \cdots, n\}$. If f and g are any two bijections from A to A then so is their composition. Consider the set of all possible bijections from A to A. These are called permutations of A, and each permutation can be considered as a way to mixup or reorder the elements of A. The set of permutations is denoted S_n and has many interesting properties and applications. Note that the identity function defined by $id(x) = x$ for all x is an element of S_n, and that for any other permutation f, $id \circ f = f$ and $f \circ id = f$. Also, S_n is closed under the operation of composition. This operation is associative, but not commutative. It is also noteworthy that the number of elements in S_n is 2^n. That follows from a comment in the earlier in the digression ★.

End of Day