Day 20: Friday, 3/31/2017
Return homework; Take questions on 6.6; Collect 6.6 homework.
Re-Announce exam on April 7.

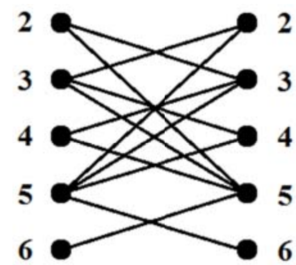
Next Section: 7.1 Relations

1. Overview
   a. We have many examples of ways that two mathematical objects can be *related*. For example, they can be equal (written $x = y$), or one number can be greater than another ($x > y$), or one set can be a subset of another ($A \subseteq B$), or one integer can be a divisor of another ($m \mid n$), or something can be an element of a set of things ($b \in B$), or one logical statement can imply another ($A \to B$) …
   b. In chapter 7 we look at this phenomenon more generally, unifying all of these examples as instances of one common concept
   c. To do so, we recognize that in each example, we can think of the relation of interest as a set of ordered pairs. For example, the > relation can be considered as the set of all (x, y) for which $x > y$. Clearly, (2,1) is IN this set, while (1,1) is NOT in the set. We can conceptualize the greater than relation entirely in terms of this set of ordered pairs.

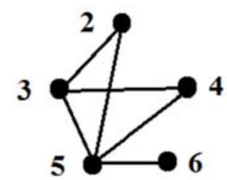
2. Definitions and Notation
   a. A relation from set A to set B is a subset of $A \times B$. That is, a relation is a set of ordered pairs (x, y) where each x is from the set A and each y is from the set B.
   b. Special case: A and B are the same set. We say "a relation on A" rather than "a relation from A to A. For example, "greater than" is a relation on $\mathbb{R}$.
   c. The domain of a relation R from A to B is the set of all the elements that can appear as first elements in the set of ordered pairs, and the range is the set of all elements that can appear as second elements. These are denoted dom(R) and range(R).
   d. In symbols, if R is a relation from A to B, the statement ($a \in \text{dom}(R)$) means $(\exists\, b \in B)(\,(a, b) \in R)$. The dom($R$) is a subset of A, but unlike the case for functions, the domain need not be *all* of A. For example, if R is the divisor relation on $\mathbb{Z}$, the domain does not include 0. We can never have $(0, n)$ in R because 0 is not a divisor of any number.
   e. Similarly, the range of R is characterized is symbols as follows. The statement ($b \in \text{range}(R)$) means $(\exists\, a \in A)(\,(a, b) \in R)$. This is a subset of B, and *like* the case for functions, the range need not be *all* of B. For example, if R is the "element of " relation from a set A to the power set $\wp(A)$, the range does not include the empty set $\emptyset$. We can never have $(a, \emptyset)$ in R because "$a \in \emptyset$" is never true.
   f. For a relation R, we write $x\, R\, y$ to mean $(x, y) \in R$, and write $x \not R\, y$ to mean $(x, y) \notin R$. This is consistent with familiar examples of relations. For example, if we think of $\in$ as the name of the "element of " relation, we write $a \in A$ and $a \notin A$.
   g. Sometimes we wish to discuss a generic abstract relation – just as we sometimes discuss a general function f without having a particular function in mind, or a generic set A

without having a specific set in mind. It is common to represent a generic relation with the symbol ~. So we write things like $x \sim y$ and $x \nsim y$ for this generic relation.

3. Arrow diagrams
   a. We can illustrate (or define) a relation with the same sort of mapping diagram we used for functions. Draw a line from a to b whenever $a \sim b$.
   b. Example. The diagram at right defines a relation from {2, 3, 4, 5, 6} to itself. This relation says $x \sim y$ iff x and y have no common divisor. Note that the definition is symmetric, and the diagram *reflects* that, so to speak.
   c. This relation is not a function in either direction



4. Graphs and Digraphs
   a. For a relation from a set A to itself, another visual representation is constructed as follows
   b. Make one dot or vertex for each element of A.
   c. Draw an arrow from x to y iff $x \sim y$.
   d. If every arrow goes in both directions, then arrows can be omitted.
   e. Example: the diagram at right represents the same example as in 3*b*.
   f. This sort of dot and line diagram is called a *graph*. If there are arrows on the lines it is a *digraph* (directional graph)
   g. If $a \sim a$ then we make a loop attached to the dot for a.



5. *xy* Plane Graphs. When A and B are both $\mathbb{R}$ or subsets of $\mathbb{R}$, all subsets of $A \times B$ are also subsets of $\mathbb{R}^2$, and so can be visualized as subsets of the *xy* plane. These can be graphed in the usual way. For example, consider the relation $\{(x, y) \in \mathbb{R}^2 \mid x + y < 1\}$. This consists of all points (x, y) satisfying the inequality $x + y < 1$. These are all the points lying below the line $x + y = 1$, and is typically illustrated as a shaded region in the *xy* plane. It is technically correct to describe the region as a relation, but usually the relations that arise naturally have a more obvious meaning as a way of relating one quantity to another.

6. Relations vs Functions: Every function from A to B is a relation from A to B, but not vice versa. Functions are a special type of relations. The set of functions from A to B is a proper subset of the set of relations from A to B.

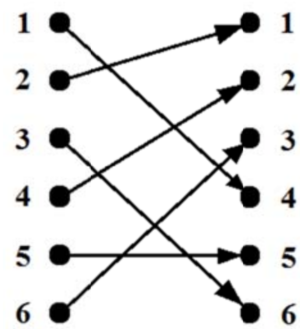
7. Time permitting: Orbits of permutations and the "in the same orbit with" relation.
   a. As we saw in some examples when we were studying induction, when $f : A \to A$ we can create a sequence by starting with an initial x, applying f, then applying f to the result, and continuing to apply f over and over: $x_0 \xrightarrow{f} x_1 \xrightarrow{f} x_2 \xrightarrow{f} x_3 \xrightarrow{f} \ldots$
   b. Suppose f is a bijection and A is a finite set. These sequences then must be cyclic – at some point a repetition must occur, because there are only finitely many x's available. But the first time that happens, the repeated value must be x_0. Otherwise, the first time it

happens you would have to find something like $x_{43} = x_{17}$. But then you would also know that $x_{42} = x_{16}$ (because $x_{43} = f(x_{42})$ and $x_{17} = f(x_{16})$ and f is injective). That contradicts the assumption that we are considering the *first* repetition. So the first repetition must be of the form $x_{26} = x_0$, and from that point on the sequence repeats exactly the same values as it had starting from the initial value of x_0.

c. So now we know, for a bijection of a finite set, any starting point leads to a cyclic sequence. That sequence is called the orbit of x_0, because it is all the points that x_0 can *visit* as you repeatedly apply f. But of course, all the elements of a cycle can be chosen as the first element, and they all generate the same cycle. So all of the element of a cycle have the same orbit.

d. So let us define a relation on A that says $x \sim y$ iff x and y have the same orbit.

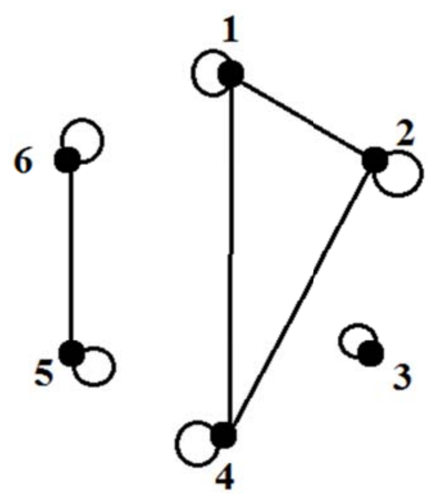
e. Example. Let A be $\{1, 2, 3, 4, 5, 6\}$. Define a bijection f as the set $\{(1, 4), (2, 1), (3, 6), (4, 2), (5, 5), (6, 3)\}$. We can also visualize f with the diagram shown at right. What are the orbits?



f. Start with initial value 1. Apply f and you reach 4. Apply f again (to 4) and you obtain 2. Apply f again and you get to 1. From that point on the cycle repeats. So one orbit of f is $1 \to 4 \to 2 \to 1$, and we see that each of the three elements is related by $\sim$ to itself and the other two elements. That is, $1 \sim 1$, $1 \sim 2$, $1 \sim 4$, $2 \sim 1$, $2 \sim 2$, $2 \sim 4$, $4 \sim 1$, $4 \sim 2$, and $4 \sim 4$.

g. Using a similar analysis we find that $3 \to 6 \to 3$ is another orbit, and $\{5\}$ is an orbit all by itself.

h. Recalling that $\sim$ is defined to hold whenever two elements are in the same orbit, we can make the diagram of this relation shown at right.



End of Day