Day 23: Tuesday, 4/11/2017
Return exams, have students arrange groups for exam solutions

Comment on Proofs from Exam

1. Scoping of variables as in the second proof. Relate to dummy variables like the index of a sum or the integration variable in a definite integral.
   a. In a proof, when you write "Let a be an element of A," that should be thought of as assigning a specific value to the variable a. The reader should be able to say "Okay, I am going to choose my a to be 6" and all of the following statements of the proof should be valid with that meaning for a. So, following the initial declaration x, the symbol a should be treated the way you treat a constant (for example $e = 2.71\ldots$) in algebra.
   b. In contrast, when you write "For all a in A, $f(a)$ is in B," that is not assigning a value to a that persists beyond the sentence. Here we are making a general statement that applies for all valid replacements of A and the particular symbol we use is irrelevant. In other words, "For all a in A, $f(a)$ is in B" has exactly the same meaning as "For all w in A, $f(w)$ is in B," or "For all α in A, $f(\alpha)$ is in B," or "For all z in A, $f(z)$ is in B."
   c. The variables that appear in the statements in (b) are not assigned specific values, as they were in part (a). In that way they function the same way as *dummy* variables in a definite integral or summation. For example, if you write in a proof that $\sum_{k=1}^{10} k = 55$, that does not define k as a constant in later steps of the proof. As with the statements in part (b), the particular letter used has no meaning: $\sum_{k=1}^{10} k = 55$ means the same things as $\sum_{m=1}^{10} m = 55$ or $\sum_{\beta=1}^{10} \beta = 55$.
   d. One way to think about these two different uses of variables involves the mathematical concept of scope. When a variable is defined (meaning that it is entered in a proof under the control of a quantifier), the assigned meaning is assumed to hold within some definite body of writing. For example, if you say in one sentence of a proof, "let $a = 3$," that assignment is assumed to be valid through the rest of the proof, unless you specifically change the meaning. For example, you might later write, "For the next part of the proof, let $a = 5$." But no such statement would be required once the proof is complete. You would not expect the reader to continue to think of a as being the number 3 for the rest of the chapter or book, for example. We use the term scope to identify the block of writing in which the assigned value of a variable is assumed not to change.
   e. In part (a), the assumed scope of the definition of a is the entire proof, or possibly a part of the proof (for example when you do a proof by cases, or an if and only if proof). On the other hand, in a quantified statement like the ones in (b), the scope is restricted to the specific statement being made.
   f. A common scope error: making a statement like the ones in (b), and assuming that a larger scope than a single sentence. For example, saying something like this: "For all $a \in \mathbb{Z}$, $a^2 + 1 > 0$. Now suppose f is the function defined by $f(x) = x^2 + 5$, for all real x. Then we get $f(a) = a^2 + 5 = a^2 + 1 + 4 > 0 + 4$. " The first mention of a is in a quantified statement, and the presumed scope for that is just that sentence. The appearance of x in the next sentence is also explicitly quantified, and the scope again is assumed to be just

that statement. So what is the meaning of *a* in the final sentence? Is it supposed to be a real number? Any integer? The reader has no way to know. Compare that with this version: "For all $n \in \mathbb{Z}$, $n^2 + 1 > 0$. Now suppose f is the function defined by $f(x) = x^2 + 5$, for all real x, and let a be an element of $\mathbb{Z}$. Then we get $f(a) = a^2 + 5 = a^2 + 1 + 4 > 0 + 4$. " Here, the phrase "let a be an element of $\mathbb{Z}$" gives a meaning to the element a that the reader will know persists throughout the following lines of the argument.

g. Another kind of error. I often see something like this in a proof. "Let a be an integer greater than 12. For all $a \in \mathbb{Z}$, $2a$ is even." This is not technically wrong if you realize that the a in the second statement has a different meaning than the one in the first. The first sentence identifies a as a particular number, and the reader is justified is thinking, "so let me see what happens if I take $a = 13$." That value does not carry over to the next statement. It would be silly to substitute 13 for a and say "For all $12 \in \mathbb{Z}$, $2 \cdot 12$ is even." But even though this is not technically incorrect, it makes the reader's job more difficult – keeping track of which meaning of a applies where. Far better to write "Let a be an integer greater than 12. For all $n \in \mathbb{Z}$, $2n$ is even, so in particular, $2a$ is even."

New Material: Section 7.2, Equivalence Relations

1. We have seen that the concept of a *relation* is quite general. But some relations (from a set A to itself) have special properties. This is similar to the way we first had a general definition of function, and then defined subclasses of functions with special properties. In this section of the text we look at relations that have three special properties. These are called equivalence relations and they are used widely in mathematics.
2. Three special properties a relation on a set A may have
   a. Reflexivity. If $a \sim a$ for all a in A, we say the relation ~ is reflexive. For example, = on $\mathbb{R}$ and $\subseteq$ on the power set of a set U are both reflexive. But < on $\mathbb{R}$ is not. In the directed graph for a reflexive relation, there is a loop attached to every element of A. In the arrow diagram, every element is paired with itself by a horizontal line.

      In symbols, a relation R from A to A is reflexive iff $(\forall\ a \in A)((a, a) \in R)$.
   b. Symmetry. If $a \sim b \to b \sim a$ for all a and b in A, we say the relation ~ is symmetric. For example, = on $\mathbb{R}$ and "disjoint from" on the power set of a set U are both reflexive. But < on $\mathbb{R}$ is not. In the digraph for a symmetric relation, every arrow goes both ways. In the arrow diagram, the entire diagram is symmetric left to right.

      In symbols, a relation R from A to A is symmetric iff $(\forall\ (a, b) \in R)((b, a) \in R)$. Equivalently, R is symmetric iff $(\forall\ a, b \in A)((a, b) \in R \to (b, a) \in R)$.

   c. Transitivity. If $(a \sim b \wedge b \sim c) \to a \sim c$ for all a, b, and c in A, we say the relation ~ is transitive. For example, = on $\mathbb{R}$ and $\subseteq$ on the power set of a set U are both transitive. So is < on $\mathbb{R}$. But "disjoint from" on the power set of a set U is not transitive. In the digraph of a transitive relation, whenever there is a linked path from a to b to c, there is also a direct path from a to c. This is like head to tail vector addition: if **a** and **b** are in the diagram, so is the vector sum **a + b**, and can be visualized in terms of completed triangles. There is no simple way to recognize transitivity in the arrow diagram of a relation.

In symbols, a relation R from A to A is transitive iff
$(\forall\ a, b, c \in A)(((a, b) \in R \ \wedge\ (b, c) \in R) \rightarrow (a, c) \in R)$.

3. A relation on a set A is called an equivalence relation iff it is reflexive, symmetric, and transitive.

4. Example: Congruence modulo n.
   a. Definition: $a \equiv b$ mod n iff $n \mid (b - a)$.
   b. Use this fact to show congruence mod n is an equivalence relation.

5. Example: Two finite sets are called equivalent if they have the same cardinality.
   a. Definition of cardinality: Card(A) = the number of elements of A. Same cardinality means the same number of elements.
   b. Verify that this is an equivalence relation.

6. Example: Row equivalence of matrices.
   a. Definition of row equivalence: For matrices A and B, we say that B is row equivalent to A iff there exists an invertible matrix P such that $B = PA$. (This is true iff it is possible to transform A into B using elementary row operations.)
   b. Verify that this is an equivalence relation.

7. Example: same image
   a. Let $f : A \rightarrow B$ be any function. Define a relation ~ on A by the rule, $a \sim b$ iff $f(a) = f(b)$.
   b. Verify that this is an equivalence relation.
   c. For the particular case $A = \mathbb{R}$ and $f(x) = x^2$ for all real x, what is the condition for $a \sim b$?
   d. Repeat question (c) with $f(x) = \sin x$.
   e. Repeat question (c) for this function: $f : \mathbb{Z} \rightarrow \mathbb{Z}$ given by $f(n)$ equals the remainder when n is divided by 7. Give several examples of elements a and b of $\mathbb{Z}$ for which $a \sim b$ or $a \nsim b$. What familiar equivalence relation is this?

8. Other Examples. In section 7.1, several relations between a set and itself are listed in the table on p 366: < on $\mathbb{R}$, = on $\mathbb{R}$, divides (|) on $\mathbb{Z}$, and $\subseteq$ on the power set of a set U. Determine for each of these relations whether it is reflexive, symmetric, and/or transitive. Which are equivalence relations?

End of Day