Day 24: Friday, 4/14/2017

Collect regular homework for section 7.2. Double check that students found the extra problems. Take questions on 7.2 before collecting papers.

New Material: Section 7.3 Equivalence Classes and Partitions

1. Overview: Last class we introduced the concept of an equivalence relation on a set *A*. This is a relation from *A* to *A* having the properties of reflexity, symmetry, and transitivity. Now we will see that any equivalence relation on *A* defines a way of grouping the elements of *A* into mutually exclusive subsets. Such a grouping is referred to as a partition. This is characterized by the properties that every element of *A* must belong to one and only one of the subsets making up the partition.

2. Equivalence classes relative to an equivalence relation.
   a. Definition: If *R* is an equivalence relation on *A* represented by the symbol ≡, for any element *a* in *A*, the equivalence class of *a* is the set of all *b* that are equivalent to *a*. In symbols we write $[a] = \{b \in A \mid b \equiv a\}$. In there is any possible ambiguity about which relation we are considering (for example if we are simultaneously considering two or more equivalence relations), we will write $R[a]$ for the equivalence class of *a*.
   b. Observation: for every *a* in *A*, $a \in [a]$. (Why?)
   c. Example: Equivalence classes of $\mathbb{Z}$ relative to the equivalence relation "congruent modulo *n*".
      - Particular case: $n = 3$
      - $[5]$ = {all integers congruent to 5 mod 3} = $\{k \in \mathbb{Z} \mid (k - 5)$ is divisible by $3\}$
      - A little experimentation will show that this is the set {… , -4, -1, 2, 5, 8, 11, … }
      - We can also derive this algebraically by recognizing that x iff $(k - 5)$ is divisible by 3 iff $k - 5 = 3t$ for some integer *t* iff $k = 5+3t$ for some integer *t*.
      - Notice that $[5] = [2]$
      - In fact, we know every integer is either congruent to 0, 1, or 2 mod 3
      - So every integer is in [0] or [1] or [3], and can be in only one of these.
      - The same properties hold for equivalence classes modulo *n* for any *n*.

3. Properties of Equivalence Classes
   a. Statement of Theorem 7:14: Let A be a nonempty set and let ~ be an equivalence relation on the set *A*. Then,
      i. For each $a \in A$, $a \in [a]$.
      ii. For each $a, b \in A$, $a \sim b$ iff $[a] = [b]$.
      iii. For each $a, b \in A$, $[a] = [b]$ or $[a] \cap [b] = \emptyset$.
      iv. Restatement of third property: For each $a, b \in A$, $[a] \cap [b] \neq \emptyset \rightarrow [a] = [b]$.

b. Proof of (i). Let $a \in A$. Since ~ is an equivalence relation, $a \sim a$. Thus, by definition of equivalence class, $a \in [a]$.

c. Proof of (ii). This is a biconditional so we have to prove two parts.

- First we prove $a \sim b \to [a] = [b]$.
  1. Assume $a \sim b$. We will show that $[a] \subseteq [b]$. To that end, let $c \in [a]$. Thus $c \sim a$. But we also know $a \sim b$. So by transitivity, $c \sim b$. The definition of equivalence class now shows $c \in [b]$. Therefore, we have proven $[a] \subseteq [b]$.
  2. Now we have actually shown that $a \sim b \to [a] \subseteq [b]$. But when $a \sim b$, by symmetry, we also know $b \sim a$, and that implies $[b] \subseteq [a]$ by what we just proved (with the roles of a and b interchanged). That is, we have shown that $a \sim b \to [b] \subseteq [a]$.
  3. Combining both results, we have shown that $a \sim b \to [a] = [b]$.
- For the converse, we prove $[a] = [b] \to a \sim b$. So assume $[a] = [b]$. We know (by part (i) of the theorem) that $a \in [a]$. Thus $a \in [b]$. By the definition of $[b]$, that means $a \sim b$, which is what we wished to show. This completes the proof of the biconditional, and establishes the part (ii) of the theorem

d. Proof of (iii). Actually we prove (iv). Assume $[a] \cap [b] \neq \emptyset$ and let $c \in [a] \cap [b]$. Thus $c \sim a$ and $c \sim b$. Applying symmetry, $c \sim a$ implies $a \sim c$. So we know $a \sim c$ and $c \sim b$, and transitivity shows $a \sim b$. Therefore, by part (ii) of the theorem, $[a] = [b]$. This shows $[a] \cap [b] \neq \emptyset \to [a] = [b]$, as desired.

4. The collection of equivalence classes.
   a. In general, given an equivalence relation on a set A, we can consider the set of all equivalence classes. This is a subset of the power set of A, and can be expressed as follows: $\{[a] \mid a \in [a]\}$.
   b. However note that the number of equivalence classes is usually less than the number of elements of A, because there may be many cases where $a \neq b$ but $[a] = [b]$.
   c. To see this in a specific example, suppose that $A = \{1, 2, 3, 4, 5, 6\}$ and the equivalence classes are $[1] = [2] = [3] = \{1, 2, 3\}$; $[4] = [5] = \{4, 5\}$, and $[6] = \{6\}$. Then the complete collection of equivalence classes is $\{[1], [4], [6]\}$. We *could* write this as {[1], [2]. [3]. [4], [5], [6]} but several of the elements in this roster are redundant.
   d. The collection of equivalence classes provides an organizational structure for A. Each equivalence class is viewed as a set of interrelated (or equivalent) elements of A. Each element of A is in one and only one of the classes. This is implied by Theorem 7.14.
   e. Put another way, the equivalence classes have these three properties:
      i. The classes are all nonempty.
      ii. Any two distinct classes are disjoint from each other
      iii. The union of all the classes is equal to A
   f. We say that these sets *partition* A. In general, any collection of subsets of A that has these properties defines a *partition* of A. Theorem 7.18 says that the for any equivalence

relation on a set *A*, the equivalence classes define a partition of *A*.

5. The book highlights these properties in the specific case of congruence modulo an integer *n*. This is done in corollary 7.16 and 7.17.

6. At the end of section 7.3, it is stated that every partition defines an equivalence relation, just as every equivalence relation defines a partition. The idea is this: given a partition, we define a relation by declaring that any two elements are equivalent if they are in the same partition set. The three properties of a partition can then be used to show that this relation is reflexive, symmetric, and transitive.

7. Time permitting, we will examine the examples at the end of the preceding set of lecture notes, and look at the equivalence classes in each case. These are items 5, 6, and 7 of the last page of the notes for the preceding lecture.

End of Day