Day 25: Tuesday, 4/18/2017
Take questions.
Collect: exam solutions, quiz problem from 7.2, regular probs from 7.3.

New topic: Section 9.1: Finite Sets

1. Equivalent sets
   a. Definition: Two sets $A$ and $B$ are *equivalent* provided there exists a bijection from $A$ to $B$.
   b. Notation and terminology: If $A$ and $B$ are equivalent we write $A \approx B$, and we say that $A$ and $B$ have the same cardinality. If the sets are not equivalent we write $A \not\approx B$, and say they have different cardinalities.
   c. The property of being equivalent is reflexive because the identity function is a bijection from any set to itself.
   d. The property of being equivalent is symmetric because a bijection always has an inverse function that is also a bijection. Thus if $f : A \to B$ is a bijection, so is $f^{-1}: B \to A$.
   e. The property of being equivalent is transitive because the composition of bijections is a bijection. Thus if $f : A \to B$ and $g$: $B \to C$ are bijections, so is $g \circ f : A \to C$.
2. The subject of set theory does not work if we permit an all encompassing universal set – a set of all sets.
   a. Implication: there is no context within which to define set equivalence as a relation from one set to another.
   b. This is the meaning of the technical note at the bottom of page 453.
   c. Why can't there be a set of all sets? If we include such a thing in our set theory, we also are forced to include a contradictory statement.
   d. The underlying problem concerns the idea that a set might be an element of itself. Not a subset – an element. Note that if there is a set of all sets, since it is itself a set, then it is an element of itself.
   e. There are many other examples: the set $X$ of all sets that do not contain the element 1. Note that $X$ does not contain the element 1, because 1 is not a set, so is not a set that does not contain 1. So $X \in X$. Another example: the set $Y$ of all infinite sets. The set $Y$ is itself infinite, because it contains as elements, the sets $\{n\}^c = \mathbb{Z} - \{n\}$ where $n$ can be any integer. So $Y \in Y$. An example of a different sort: the set of all abstract concepts.
   f. Of course, there are also many sets that are *NOT* elements of themselves. $\{a, b, c\}$ is a set, and it has three elements, namely the letters $a$ , $b$, and $c$. But it does not contain the set $\{a, b, c\}$. So this is a set that is NOT an element of itself.
   g. So, if there is a set $\mathbf{\Omega}$ of all sets, let's define the set $P = \{A \in \mathbf{\Omega} \mid A \notin A\}$.
   h. Now either $P \in P$, or $P \notin P$. But if $P \in P$, then by definition, $P$ is a set that is not an element of itself, and by definition of $P$, we must conclude $P \notin P$. That is a contradiction. At the same time, if $P \notin P$, then $P$ is a set that is not an element of itself, and by definition of $P$, we must conclude $P \in P$. Another contradiction. In other words, the existence of the set $P$ forces our system of logic to include a contradiction, which in turn implies that every statement in the system is true.

3. Finite Set Definition and properties
   a. Defintion: Let $k \in \mathbb{N}$. We define $\mathbb{N}_k = \{j \in \mathbb{N} \mid j \leq k\} = \{1, 2, 3, \cdots, j\}$.
   b. Definition: A set $A$ is a *finite set* provided that $A = \emptyset$ or $A \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$. $A$ is an infinite set provided that $A$ is not a finite set.
   c. If $A \approx \mathbb{N}_k$ (for some natural number $k$) then we say that $A$ has cardinality $k$ and write card($A$) = $k$. If $A$ is empty then we say that $A$ has cardinality 0.
   d. Theorem 9.3 says: Any set equivalent to a finite set $A$ is also finite and has the same cardinality as $A$. This follows because the composition of bijections is a bijection.
   e. For all of the above to make sense, we need to know that a set cannot be equivalent to two different $\mathbb{N}_k$'s. Although this seems obvious for finite sets, we can (and should) formulate a rigorous proof. By the same reasoning as in theorem 9.3, it suffices to show that there is no bijection between $\mathbb{N}_j$ and $\mathbb{N}_k$ if $j \neq k$. This follows from Theorem 9.6.
   f. Lemma 9.4: inserting one new element in a finite set $A$ results in a finite set having cardinality one greater. That is, the cardinality of the new set equals card($A$) + 1.
   g. Lemma 9.5: every subset of $\mathbb{N}_k$ has cardinality $\leq k$.
   h. Theorem 9.6: If $S$ is a finite set and $A \subseteq S$, then $A$ is a finite set and card($A$) ≤ card($S$).
   i. Corollary 9.7: If $A$ is a finite set and $x \in A$, then $A - \{x\}$ is a finite set and card($A - \{x\}$) = card($A$) – 1. This can be proved by applying lemma 9.4, letting $A - \{x\}$ in the current theorem play the role of $A$ in the lemma. Details are left as an exercise.
   j. Note that we now see that card($\mathbb{N}_1$) < card($\mathbb{N}_2$) < card($\mathbb{N}_3$) < ⋯ , so no two distinct $\mathbb{N}_k$'s can have equal cardinality. (Rigorous proof possible using induction.) This shows that a finite set can only be equivalent to one $\mathbb{N}_k$, and so cardinality is a legitimate function.
   k. Corollary 9.8: A finite set is not equivalent to any of its proper subsets. Sketch of proof: if the finite set is empty, it *has* no proper subsets. Otherwise, if $A$ is a proper subset of $B$ then there is some element $x$ that is in $B$ but not in $A$. That makes $A$ a subset of $B - \{x\}$. Thus card($A$) ≤ card($B - \{x\}$) = card($B$) – 1 by two prior results. Thus card($A$) < card($B$).
4. Pigeon Hole Principle: Let $A$ and $B$ be finite sets with card($A$) > card($B$). Then for any function $f : A \to B$, there must exist $a_1 \neq a_2$ in $A$ with $f(a_1) = f(a_2)$. (That is, $f$ is not an injection.)
   a. Example: In any group of 400 people, two must have the same birthday. Proof: let $A$ be a group of 400 people. Let $B$ be the set of calendar days: Jan 1, Jan 2, ⋯ , Dec 31, including Feb 29. Let $f : A \to B$ be the birthday function. Since card(A) = 400 and card(B) = 366 < 400, the PHP applies, and there exist people $p$ and $q$ in the group with $f(p) = f(q)$. That is, $p$ and $q$ have the same birthday.
   b. Some applications are surprising and profound.
      i. in any group of six people there are either three mutual friends or three mutual strangers.
      ii. The decimal expansion of any rational number is either finite or eventually is a repeating decimal
      iii. If $f$ is a bijection on a finite set, for any initial $x_0$, the recursive sequence $x_0, f(x_0), f(f(x_0)), f(f(f(x_0)))$ , ⋯ is cyclic.

c. Prove PHP by proving the contrapositive: if there exists injection $f : A \to B$, then card($A$) $\leq$ card($B$). Main idea is that the range $R = f(A)$ is a subset of $B$ and that $f$ is a bijection of $A$ onto $R$. Thus card ($A$) = card($R$) $\leq$ card($B$).

End of Day