

Day 25: Tuesday, 4/18/2017

Take questions.

Collect: exam solutions, quiz problem from 7.2, regular probs from 7.3.

New topic: Section 9.1: Finite Sets

1. Equivalent sets

- a. Definition: Two sets A and B are *equivalent* provided there exists a bijection from A to B .
 - b. Notation and terminology: If A and B are equivalent we write $A \approx B$, and we say that A and B have the same cardinality. If the sets are not equivalent we write $A \not\approx B$, and say they have different cardinalities.
 - c. The property of being equivalent is reflexive because the identity function is a bijection from any set to itself.
 - d. The property of being equivalent is symmetric because a bijection always has an inverse function that is also a bijection. Thus if $f: A \rightarrow B$ is a bijection, so is $f^{-1}: B \rightarrow A$.
 - e. The property of being equivalent is transitive because the composition of bijections is a bijection. Thus if $f: A \rightarrow B$ and $g: B \rightarrow C$ are bijections, so is $g \circ f: A \rightarrow C$.
2. The subject of set theory does not work if we permit an all encompassing universal set – a set of all sets.
- a. Implication: there is no context within which to define set equivalence as a relation from one set to another.
 - b. This is the meaning of the technical note at the bottom of page 453.
 - c. Why can't there be a set of all sets? If we include such a thing in our set theory, we also are forced to include a contradictory statement.
 - d. The underlying problem concerns the idea that a set might be an element of itself. Not a subset – an element. Note that if there is a set of all sets, since it is itself a set, then it is an element of itself.
 - e. There are many other examples: the set X of all sets that do not contain the element 1. Note that X does not contain the element 1, because 1 is not a set, so is not a set that does not contain 1. So $X \in X$. Another example: the set Y of all infinite sets. The set Y is itself infinite, because it contains as elements, the sets $\{n\}^c = \mathbb{Z} - \{n\}$ where n can be any integer. So $Y \in Y$. An example of a different sort: the set of all abstract concepts.
 - f. Of course, there are also many sets that are *NOT* elements of themselves. $\{a, b, c\}$ is a set, and it has three elements, namely the letters a , b , and c . But it does not contain the set $\{a, b, c\}$. So this is a set that is NOT an element of itself.
 - g. So, if there is a set Ω of all sets, let's define the set $P = \{A \in \Omega \mid A \notin A\}$.
 - h. Now either $P \in P$, or $P \notin P$. But if $P \in P$, then by definition, P is a set that is not an element of itself, and by definition of P , we must conclude $P \notin P$. That is a contradiction. At the same time, if $P \notin P$, then P is a set that is not an element of itself, and by definition of P , we must conclude $P \in P$. Another contradiction. In other words, the existence of the set P forces our system of logic to include a contradiction, which in turn implies that every statement in the system is true.

3. Finite Set Definition and properties

- a. Definition: Let $k \in \mathbb{N}$. We define $\mathbb{N}_k = \{j \in \mathbb{N} \mid j \leq k\} = \{1, 2, 3, \dots, j\}$.
 - b. Definition: A set A is a *finite set* provided that $A = \emptyset$ or $A \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$.
 A is an infinite set provided that A is not a finite set.
 - c. If $A \approx \mathbb{N}_k$ (for some natural number k) then we say that A has cardinality k and write $\text{card}(A) = k$. If A is empty then we say that A has cardinality 0.
 - d. Theorem 9.3 says: Any set equivalent to a finite set A is also finite and has the same cardinality as A . This follows because the composition of bijections is a bijection.
 - e. For all of the above to make sense, we need to know that a set cannot be equivalent to two different \mathbb{N}_k 's. Although this seems obvious for finite sets, we can (and should) formulate a rigorous proof. By the same reasoning as in theorem 9.3, it suffices to show that there is no bijection between \mathbb{N}_j and \mathbb{N}_k if $j \neq k$. This follows from Theorem 9.6.
 - f. Lemma 9.4: inserting one new element in a finite set A results in a finite set having cardinality one greater. That is, the cardinality of the new set equals $\text{card}(A) + 1$.
 - g. Lemma 9.5: every subset of \mathbb{N}_k has cardinality $\leq k$.
 - h. Theorem 9.6: If S is a finite set and $A \subseteq S$, then A is a finite set and $\text{card}(A) \leq \text{card}(S)$.
 - i. Corollary 9.7: If A is a finite set and $x \in A$, then $A - \{x\}$ is a finite set and $\text{card}(A - \{x\}) = \text{card}(A) - 1$. This can be proved by applying lemma 9.4, letting $A - \{x\}$ in the current theorem play the role of A in the lemma. Details are left as an exercise.
 - j. Note that we now see that $\text{card}(\mathbb{N}_1) < \text{card}(\mathbb{N}_2) < \text{card}(\mathbb{N}_3) < \dots$, so no two distinct \mathbb{N}_k 's can have equal cardinality. (Rigorous proof possible using induction.) This shows that a finite set can only be equivalent to one \mathbb{N}_k , and so cardinality is a legitimate function.
 - k. Corollary 9.8: A finite set is not equivalent to any of its proper subsets. Sketch of proof: if the finite set is empty, it *has* no proper subsets. Otherwise, if A is a proper subset of B then there is some element x that is in B but not in A . That makes A a subset of $B - \{x\}$. Thus $\text{card}(A) \leq \text{card}(B - \{x\}) = \text{card}(B) - 1$ by two prior results. Thus $\text{card}(A) < \text{card}(B)$.
4. Pigeon Hole Principle: Let A and B be finite sets with $\text{card}(A) > \text{card}(B)$. Then for any function $f: A \rightarrow B$, there must exist $a_1 \neq a_2$ in A with $f(a_1) = f(a_2)$. (That is, f is not an injection.)
- a. Example: In any group of 400 people, two must have the same birthday. Proof: let A be a group of 400 people. Let B be the set of calendar days: Jan 1, Jan 2, \dots , Dec 31, including Feb 29. Let $f: A \rightarrow B$ be the birthday function. Since $\text{card}(A) = 400$ and $\text{card}(B) = 366 < 400$, the PHP applies, and there exist people p and q in the group with $f(p) = f(q)$. That is, p and q have the same birthday.
 - b. Some applications are surprising and profound.
 - i. in any group of six people there are either three mutual friends or three mutual strangers.
 - ii. The decimal expansion of any rational number is either finite or eventually is a repeating decimal
 - iii. If f is a bijection on a finite set, for any initial x_0 , the recursive sequence $x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), \dots$ is cyclic.

- c. Prove PHP by proving the contrapositive: if there exists injection $f: A \rightarrow B$, then $\text{card}(A) \leq \text{card}(B)$. Main idea is that the range $R = f(A)$ is a subset of B and that f is a bijection of A onto R . Thus $\text{card}(A) = \text{card}(R) \leq \text{card}(B)$.

End of Day