Day 26: Friday, 4/21/2017

Return Homework and model solution. Take questions on 9.1. Collect 9.1 homework and final quiz problem.

Lecture: Section 9.2: Countable Sets

1. Recap of finite and infinite sets
   a. equivalence and equal cardinality
   b. Finite defined as equivalence to one of the special sets $\mathbb{N}_k$.
   c. Infinite defined as not finite
   d. How do we prove a set is infinite? For example, to show $\mathbb{Z}$ is infinite, we must show that $\mathbb{Z} \not\approx \mathbb{N}_k$ for every k.
   e. An alternative: use the contrapositive of the theorem that no finite set is equivalent to a proper subset: If A *is* equivalent to a finite subset then A is not infinite
   f. Here we are avoiding a kind of intuitive approach to the infinite. We don't say $\mathbb{Z}$ is infinite because it is obvious that it never ends. Rather, we argue strictly from the definition and from results derived from the definition.
2. Example: Show that $\mathbb{N}$ is equivalent to its proper subset {positive even integers}. $f(n) = 2n$ is a bijection. Thus $\mathbb{N}$ is an infinite set.
3. Theorem 9.10 provides another way to establish infinitude
   a. 1st conclusion: If A is infinite and $A \approx B$ then B is infinite.
   b. Proof: Argue by contradiction. Suppose B is not infinite. Then $B \approx \mathbb{N}_k$ for some $k \in \mathbb{N}$. By transitivity, this implies $A \approx \mathbb{N}_k$ for the same k. But then A would be finite, a contradiction. So B cannot be finite.
   c. 2nd conclusion: If A is infinite and $A \subseteq B$ then B is infinite.
   d. Proof: Left as an exercise on the homework assignment
   e. These results give us two more ways to show a set is infinite – either by showing it is equivalent to one we already know is infinite, or by showing that it contains an infinite set. As an example of the second, since we already know that $\mathbb{N}$ is an infinite set, so are the sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, because these sets each contain $\mathbb{N}$ as a subset.
4. Infinite Cardinalities
   a. In the next section we see that there can be infinite sets of unequal cardinality. For example, we will see that there is no bijection from $\mathbb{N}$ onto $\mathbb{R}$, and therefore that $\mathbb{N} \not\approx \mathbb{R}$.
   b. OTOH, we will also see that there *does* exist a bijection from $\mathbb{N}$ onto $\mathbb{Q}$, and therefore that $\mathbb{N} \approx \mathbb{Q}$.
   c. Although in each of these cases the sets are infinite, nevertheless that doesn't necessarily mean that they have the same cardinality. Accordingly, we have to recognize that there may be many (perhaps infinitely many) different *sizes* for infinite sets. And it turns out that there are important implications of this fact. Therefore, it makes sense to categorize sets by cardinality – grouping together sets that are all equivalent to one another. This is

essentially the same idea as formulating equivalence classes, always remembering that set equivalence is not a *relation* as we have defined it.

d. The natural numbers provide a way to define the sizes of finite sets, and for each natural number k we have a *standard* instance of a set of that size, namely $\mathbb{N}_k$.
e. Now we *invent* a new collection of numbers, the infinite cardinals, by defining for each a standard instance of a set of that size.
f. For the first infinite cardinal, the standard set is $\mathbb{N}$.

5. Countably infinite sets
   a. Consider all the sets that are equivalent to $\mathbb{N}$. These sets are said to be countably infinite. We know they are infinite by theorem 9.10, and we use the adjective *countably* to remind us that they have the same cardinality as the counting numbers, $\mathbb{N}$.
   b. We introduce the symbol $\aleph_0$, pronounced aleph nought or aleph null or aleph zero, to represent the common cardinality of all the sets that are equivalent to $\mathbb{N}$.
   c. Definition: The cardinality of $\mathbb{N}$ is denoted $\aleph_0$. Accordingly, we can describe the size of $\mathbb{N}$ by saying card($\mathbb{N}$) = $\aleph_0$.
   d. Definitions:
      i. A set A is countably infinite iff $A \approx \mathbb{N}$. In this case we say card(A) = $\aleph_0$. Sometimes the term *denumerable* is used as a synonym for countably infinite.
      ii. A set A is *countable* if it is either finite or countably infinite.
      iii. A set A is *uncountable* if it is infinite but not countably infinite. ($\mathbb{R}$ is uncountable because $\mathbb{R}$ is an infinite set but $\mathbb{R} \not\approx \mathbb{N}$.)

6. Examples of Countably Infinite Sets
   a. The set $\mathbb{N}$ of natural numbers is countably infinite because $\mathbb{N} \approx \mathbb{N}$.
   b. The set $\mathbb{Z}$ of integers numbers is countably infinite. Book constructs a bijection $f : \mathbb{N} \to \mathbb{Z}$, and gives a detailed proof that this function is injective and surjective. Conceptually, f maps the even natural numbers to the nonnegative integers, and maps the odd natural numbers to the negative integers.
   c. The set of positive rationals is countably infinite. The book gives one demonstration of this. We will see another in a few minutes. This result seems a bit more surprising than the two prior examples, because the rational numbers are so dense: between any two rational numbers a and b there are infinitely many other rational numbers, including the numbers 1/2 way from a to b, 1/3 of the way, 1/4 of the way, etc. In fact, for any rational number x between 0 and 1, $a + x(b - 1)$ is a rational number between a and b that is "x of the way" from a to b.

7. Countably infinite sets can be listed roster fashion
   a. Let A be a countably infinite set. Then there is a bijection $f : \mathbb{N} \to A$. We can use this to create a roster of the elements of A.
   b. Define the symbols $a_1 = f(1)$, $a_2 = f(2)$, etc.

c. Because f is a bijection, every element of A is identified as a_k for some k in $\mathbb{N}$, and no two of the symbols represent the same element of A.
d. Therefore, $A = \{a_k \mid k \in \mathbb{N}\} = \{a_1, a_2, a_3, \cdots\}$.
e. Whenever we are dealing with a countably infinite set, we can assume that the elements can be listed in this fashion.
f. It is sometimes said in this situation that A is indexed by the natural numbers.
g. NOTE: not all infinite sets can be listed in this way. For example, there is no way to index the reals using natural numbers. No matter how you list the reals, you run out of natural numbers to use as subscripts before you have labeled all the reals. Put another way, no function from $\mathbb{N}$ to $\mathbb{R}$ is surjective.

8. Combinations of Countably Infinite Sets
   a. There are several ways of combining 2 or more countable sets with the resulting set still being countable.
   b. Theorem 9.15: If A is countably infinite and $x \notin A$, then $A \cup \{x\}$ is countably infinite.
   Proof: If x is an element of A, then $A \cup \{x\} = A$ so there is nothing to prove. Therefore, we consider the case that $x \notin A$. Since A is countable, we can list it roster fashion as $\{a_1, a_2, a_3, \cdots\}$. Thus, $A \cup \{x\} = \{x, a_1, a_2, a_3, \cdots\}$. We define a bijection f from this set to $\mathbb{N}$ as follows: $f(x) = 1$; $f(a_k) = k + 1$ for $k \in \mathbb{N}$.
   c. Theorem 9.16: If A is countably infinite and B is finite, then $A \cup B$ is countably infinite.
   Proof: This is an exercise in the homework.
   d. Theorem 9.17 extends the prior result to the union of two disjoint countably infinite sets. More generally, the union of a countable collection of countably infinite sets is countably infinite. This is an extra exercise in the homework.
   e. A related result says that the Cartesian product of finitely many countably infinite sets is countably infinite. This is also an exercise.
   f. As a special instance of this, let us see in detail that $\mathbb{N} \times \mathbb{N}$ is countably infinite, by constructing a bijection $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. One way we can do that is to describe a systematic way to count the elements of $\mathbb{N} \times \mathbb{N}$. That means we point to each element of $\mathbb{N} \times \mathbb{N}$ in turn, saying the counting numbers as we proceed. For example, in the diagram at right, you can see that the ordered pairs of $\mathbb{N} \times \mathbb{N}$ are arranged in a pattern which, if continued, will list each element exactly once. Point first to the yellow (1,1), and say "one". Then point to the red cells, (2,1) and (1,2), saying "two" and "three".

(1,1)	(1,2)	(1,3)	(1,4)	⋯
(2,1)	(2,2)	(2,3)	(2,4)	⋯
(3,1)	(3,2)	(3,3)	(3,4)	⋯
(4,1)	(4,2)	(4,3)	(4,4)	⋯
⋮	⋮	⋮	⋮	⋱

   Then point to the green cells, (3.1), (2,2) and (1,3), saying "four", "five", and "six". As we carry out this process, the ordered pair you point to is an element x in $\mathbb{N} \times \mathbb{N}$, and the spoken word is the image $f(x)$ in $\mathbb{N}$. For example, when we point to (2,2) and say "five", that means we have defined $f(2,2) = 5$. To see that this f is a bijection, we have to convince ourselves that we point to each element of $\mathbb{N} \times \mathbb{N}$ exactly once, and we say each element of $\mathbb{N}$ exactly once. On one level, you can see this conceptually from the diagram. For greater rigor, we can actually determine a formula for our function. This is discussed in exercise 9.

9. Subsets of countable sets
   a. Theorem 9.19: Every subset of $\mathbb{N}$ is countable. Book gives a partial proof, and leaves the rest to an exercise. Basic idea of proof is to count the elements of *B* in increasing order to define a bijection from *B* to $\mathbb{N}$.
   b. Corollary 9.20: Every subset of a countable set is countable. Same idea as 9.19 but replacing $\mathbb{N}$ with an arbitrary countable set. Proof is again left as an exercise.
   c. Corollary: every subset of a countably infinite set is either finite or countably infinite.
   d. Implication: $\aleph_0$ is the smallest infinite cardinal.
10. More examples of countably infinite sets
    a. The set of positive rationals $\mathbb{Q}^+$ is countably infinite. **Proof:** Observe first that $\mathbb{Q}^+$ contains the infinite set $\mathbb{N}$, and therefore by Theorem 9.10, $\mathbb{Q}^+$ is infinite. To show that $\mathbb{Q}^+$ is countable, define a function $f: \mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$ as follows. Let *q* be any element of $\mathbb{Q}^+$. Then *q* can be uniquely expressed as a fraction *a*/*b* where *a* and *b* are natural numbers with no common prime divisors. We define $f(q) = (a, b)$. This is an injection. If we redefine it as a mapping from $\mathbb{Q}^+$ to the image $f(\mathbb{Q}^+)$, then *f* becomes a bijection. This shows that $\mathbb{Q}^+$ is equivalent to a subset of the countable set $\mathbb{N} \times \mathbb{N}$, . This subset is countable by Corollary 9.20, and therefore $\mathbb{Q}^+$ is countable.
    b. The set $\mathbb{Q}$ of rationals is countably infinite. Proof: As before, we first observe that $\mathbb{Q}$ is an infinite set. To show it is countable, denote the set of negative rational numbers by $\mathbb{Q}^-$. The function $f(x) = -x$ is a bijection between $\mathbb{Q}^+$ and $\mathbb{Q}^-$, so $\mathbb{Q}^-$ is countable. Then we can express $\mathbb{Q} = \mathbb{Q}^+ \cup \{0\} \cup \mathbb{Q}^-$. This is the union of three countable sets, and so $\mathbb{Q}$ is countable by Theorem 9.17.
    c. Any infinite subset of $\mathbb{Q}$ is countably infinite. In particular, for any real numbers *a* and *b* with $a < b$, the rational elements of $[a, b]$ comprise a countably infinite set.
11. Looking forward
    a. As I have stated, there are infinite sets that are not countably infinite
    b. An example is $\mathbb{R}$.
    c. So we know there are at least two different infinite cardinals, $\aleph_0$ and card($\mathbb{R}$).
    d. This leads to several additional questions about infinite cardinals.
       i. How many different infinite cardinals are there? Infinitely many?
       ii. Is there a largest infinite cardinal?
       iii. Is card($\mathbb{R}$) the next infinite cardinal after $\aleph_0$? That is, is there an infinite cardinal that is greater than $\aleph_0$ but less than card($\mathbb{R}$)?
    e. We will see that two of these questions can be answered definitively using ideas at our disposal. The fourth question is much much more difficult. Care to guess which is the hard question?

End of Day