Day 3: Tuesday, 1/24/2017
Homework questions; collect homework

Chapter 2 overview
1. Chapter 2 provides an overview of symbolic logic
2. Although most math knowledge is stated in a natural language (like English, German, Arabic, etc), these languages often include ambiguities – multiple valid interpretations.
3. Example: A real number is positive. Does this mean *Every real number is positive* or *Some real number is positive*? In English, it can mean different things in different contexts. Consider these statements
   - *A member of this team is a thief.*
   - *A penny saved is a penny earned.*

   Usually, we would interpret the first statement to mean *Someone on this team is a thief* and the second to mean *Every penny saved is a penny earned*.
4. Symbolic logic avoids such ambiguities – it has carefully formulated rules for creating and interpreting statements.
5. We still use everyday language, and there are conventions for translating into logic. For example, "A real number is positive" is understood to mean "Every real number is positive."
6. Symbolic logic also makes it easier to interpret and manipulate complicated compound statements in logic, using essentially algebraic procedures.

Outline of ideas in chapter 2
1. Logical Operators and Compound statements
   a. Conjunction; "and"; $P \wedge Q$
   b. Disjunction; "or", $P \vee Q$ (inclusive: P or Q or both)
   c. Negation; "not", $\neg P$
   d. Implication; "implies", $P \to Q$ (also written *if P then Q*)
2. These have conceptual meanings and can be expressed in many ways in English (or other languages)
3. For the purpose of clarity, these operators are *DEFINED* explicitly
   a. Truth table defines a compound statement explicitly for every possible combination of truth or falsity of the statements from which it is formed (P and Q)
   b. The truth table is intended to be consistent with your conception of the operators
   c. If not, then you have to amend your conception, or work in an alternate mathematical reality. That is, play by a different set of rules, recognizing that you will end with a different set of conclusions.
   d. This is particularly true for the implication operator.

P	$\neg P$
T	F
F	T

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

4. Truth Tables for more complicated statements
   a. Example: Truth table for $\neg(P \vee Q)$
   b. One row needed for every possible combination of truth and falsity of constituent statements.
   c. This is derived, not defined knowledge
   d. Can be used to prove equivalence of logical statements. For example, $\neg(P \vee Q)$ is logically equivalent to $(\neg P) \wedge (\neg Q)$. This is like an algebraic identity such as $x^2 - y^2 = (x + y)(x - y)$. Whenever we see an instance of $\neg(P \vee Q)$ we can replace it with $(\neg P) \wedge (\neg Q)$ if we wish, and vice versa.
   e. Biconditional operator definition: $P \leftrightarrow Q$ means $(P \rightarrow Q) \wedge (Q \rightarrow P)$. Means "if and only if" or "is equivalent to".
   f. Because this is defined in terms of the basic operators, we can derive its truth table (as opposed to defining it).

P	Q	$P \vee Q$	$\neg(P \vee Q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

5. Tautologies and Contradictions: some compound statements are true for every row of the truth table, or false for every row. Called tautologies or contradictions, respectively.

6. Converses and Contrapositives.
   a. The statement $P \rightarrow Q$ has two related statements:
      i. The converse $Q \rightarrow P$ and
      ii. The contrapositive $(\neg Q) \rightarrow (\neg P)$
   b. From the truth of $P \rightarrow Q$ we can infer nothing about the truth of the converse. In fact, every possible combination of T and F for $P \rightarrow Q$ and its converse is possible, EXCEPT, it is not possible for both $P \rightarrow Q$ and $Q \rightarrow P$ to be false. This is a nonintuitive consequence of the truth table definition of $P \rightarrow Q$.
   c. The contrapositive of $P \rightarrow Q$ is equivalent to $P \rightarrow Q$. That means, if we want to prove $P \rightarrow Q$ it is valid to prove instead the contrapositive. This can be proved using truth tables. Proving the contrapositive is sometimes more convenient.
   d. Example 1: P: It is raining. Q: There are clouds in the sky. Here, from the truth table definition of *implies* we can verify both
      $P \rightarrow Q$ (ie (It is raining) $\rightarrow$ (There are clouds in the sky)) and
      $(\neg Q) \rightarrow (\neg P)$ (ie (There are not clouds in the sky) $\rightarrow$ (It is not raining)).
   e. Example 2: P: There are clouds in the sky. Q: TIt is raining. Here we can verify both $P \rightarrow Q$ and $(\neg Q) \rightarrow (\neg P)$ are false.
      ie (There are clouds in the sky) does not imply (It is raining) and
      (It is not raining) does not imply (There are not clouds in the sky).

7. More logical laws
    a. There are several essentially algebraic laws for manipulating statements. Examples:
        i. $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (distributive law)
        ii. $\neg (P \vee Q) \equiv (\neg P) \wedge (\neg Q)$
    b. There are other equivalences that don't have this algebraic flavor. Examples:
        i. $(P \rightarrow Q) \equiv (\neg P \vee Q)$
        ii. $\neg(P \rightarrow Q) \equiv (P \wedge \neg Q)$
    c. These can be proved using either truth tables, or via algebraic derivations with chains of equivalences. For example, we can derive f from the preceding results as follows

$(P \rightarrow Q) \equiv (\neg P \vee Q)$ (given by e above)
$\neg(P \rightarrow Q) \equiv \neg (\neg P \vee Q)$ (negate both sides)
$\neg(P \rightarrow Q) \equiv (\neg\neg P \wedge \neg Q)$ (distribute $\neg$ using c above)
$\neg(P \rightarrow Q) \equiv (P \wedge \neg Q)$ (can prove $\neg\neg P \equiv P$ using a truth table)

8. Variables: Formalizes a comment from last time: when you introduce a variable in a mathematical argument, you have to specify the set of possible replacements. Book's definition on p. 54 calls this the universal set for that variable.

9. Recall that a sentence with a variable is not a statement: $x > 1$ is neither true nor false without knowing what x is. But we can specify a set of possible replacements in such a way that every replacement makes $x > 1$ a statement. For example, let the universal replacement set by the real numbers. OTOH, since > is not defined for points in the *xy* plane, if we make that the replacement set, $x > 1$ does not become a statement for any replacement. This leads to the concept of an open sentence, which is an instance of the first case – a sentence with one or more variables so that every allowed specification of values for the variables produces a statement. In this context, we define the truth set to be the replacements that make the sentence true.

10. Sets and set notation
    a. As suggested by the preceding remarks, set notation and concepts are going to be essential for proofs
    b. We will study sets in depth in chapter 5. For now I will assume you are familiar with set notation {} and $\in$ and subsets and equality and the empty set. These are defined in the reading.
    c. Set builder notation. Example $\{x \in \mathbb{Z} \mid x > 10\}$ has the general form $\{x \in U \mid \text{open sentence in } x\}$. This notation is defined to mean the truth set within U of the open sentence. Thus $\{x \in \mathbb{Z} \mid x > 10\}$ means the set of integers which are greater than 10.

11. Quantifiers

a. Example. This is a statement "For every real x, $x^2 + 1 > 0$". In essence it is asserting a true or false statement about the truth set of the open sentence $x^2 + 1 > 0$. The opening clause is said to *quantify* the variable x in the statement. As written, the statement is true. OTOH, "For every real x, $x^2 + 1 > 1$" is also a statement, but it is false. The real number 0 is not in the truth set. That is, 0 is a real number but for $x = 0$ it is not true that $x^2 + 1 > 1$.

b. Example. This is a statement "There exists a real x for which $x^2 + 1 = 1$".
This says that the truth set (in the reals) of $x^2 + 1 = 1$ is not the empty set. That is true or false, and in fact is true, because 0 is an element of the truth set. OTOH, this is also a statement: "There exists a real x for which $x^2 + 1 = 0$", but it is a false statement.

c. The first example phrase (For every real) is referred to as universal quantification. It can be written in words this way: For all $x \in U \ldots$ and in symbolic logic as $\forall\, x \in U \ldots$.
Our first example can be written "$(\forall\, x \in \mathbb{R})(x^2 + 1 > 0)$" The symbol $\forall$ is called the universal quantifier.

d. The second example phrase (There exists a real) is called existential quantification. We write in symbols "$(\exists\, x \in \mathbb{R})(x^2 + 1 = 1)$". The symbol $\exists$ is called the existential quantifier.

e. We can combine quantification for several variables:
   i. $(\exists\, x \in \mathbb{R})(\forall\, y \in \mathbb{R})(xy = 0)$ is true
   ii. $(\exists\, x \in \mathbb{R})(\forall\, y \in \mathbb{R})(x + y = 0)$ is false
   iii. $(\forall\, x \in \mathbb{R})(\exists\, y \in \mathbb{R})(x + y = 0)$ is true

f. Negations of quantified statements follow a general rule of changing $\exists$'s to $\forall$'s and $\forall$'s to $\exists$'s and then negating the final statement. For example:
   i. $\neg(\exists\, x \in \mathbb{R})(x^2 + 1 = 0) \equiv (\forall\, x \in \mathbb{R})(x^2 + 1 \neq 0)$
   ii. $\neg(\exists\, x \in \mathbb{R})(\forall\, y \in \mathbb{R})(xy = 0) \equiv (\forall\, x \in \mathbb{R})(\exists\, y \in \mathbb{R})(xy \neq 0)$

End of Day