Day 5: Tuesday, 1/31/2017

Return chapter 2 hw; Comment: both assignments really look great. It is refreshing to see every student in class adhering to the homework guidelines from the very first assignment.

Take questions on homework. Collect ch 2 quiz problems + regular hw on chapter 3.

New Topic: Chapter 4: Induction

1. Overview
   a. This is a proof structure for proving a statement of the form $(\forall\, n \in \mathbb{N})(P(n))$
   b. First show directly that $P(1)$ is true.
   c. Then prove this: $(\forall\, n \in \mathbb{N})(P(n) \rightarrow P(n+1))$.
   d. The combination of both steps proves $(\forall\, n \in \mathbb{N})(P(n))$
2. Example: Show using the product rule that $(\forall\, n \in \mathbb{N})((x^n)' = nx^{n-1})$.
   a. Note that $P(n)$ is the open sentence $(x^n)' = nx^{n-1}$
   b. Setting $n = 1$ produces the statement $(x^1)' = 1x^0$. In other words, $P(1)$ is the statement $x' = 1$. We know this is true from calculus. Thus $P(1)$ is true.
   c. Now we want to prove: *If* $P(n)$ is true, *then* $P(n-1)$ is true.
   d. In words: *If* the power rule holds with exponent n then it also holds with exponent $n+1$.
   e. We prove this via a direct proof: Assume the power rule holds with exponent n. That is, we assume that $(x^n)' = nx^{n-1}$. Then we can differentiate x^{n+1} using the product rule, as follows:
   $$(x^{n+1})' = (x \cdot x^n)' = x' \cdot x^n + x \cdot (x^n)'.$$
   In the first term, we know $x' = 1$. For the second, by our starting assumption we know that $(x^n)' = nx^{n-1}$. Substituting those two results into the prior equation, we obtain
   $$(x^{n+1})' = 1 \cdot x^n + x \cdot (nx^{n-1}) = x^n + nx^n = (1+n)x^n.$$
   But this is the power rule with an exponent of $n+1$, which is what we wished to show. Thus we have proved $P(n)$ implies $P(n+1)$.
   f. These two steps complete the induction proof, and so show that the product rule holds for all n in $\mathbb{N}$.

3. Rationale for induction proofs
   a. What is the truth set for the open sentence $(P(n))$?
   b. This is a subset of $\mathbb{N}$.
   c. It contains 1 so it is not an empty set.
   d. For each element n in the truth set, $n+1$ is also in the truth set.
   e. This shows that the truth set must actually equal $\mathbb{N}$.
   f. The book defines an inductive set to be a subset of $\mathbb{Z}$ that is closed under adding 1. That is, if n is in the set so is $n + 1$. It is intuitively clear that once you know a particular element k is in an inductive set, so is every integer that comes after k. Our proof by induction showed that the truth set of $P(n)$ is an inductive set, and contains 1. But that means it contains every element following 1, and so is actually $\mathbb{N}$.

4. Principle of Mathematical Induction

   a. as stated in text:

> **The Principle of Mathematical Induction**
> If T is a subset of $\mathbb{N}$ such that
>
> **1.** $1 \in T$, and
>
> **2.** For every $k \in \mathbb{N}$, if $k \in T$, then $(k+1) \in T$,
>
> then $T = \mathbb{N}$.

   b. Restated: For a subset T of $\mathbb{N}$, if
      1. $1 \in T$
      2. $(\forall\, n \in \mathbb{N})(n \in T \rightarrow n+1 \in T)$

      then $T = \mathbb{N}$.

5. Can we prove the principle of mathematical induction?

   a. This is subtle. In order to prove it we need a mathematical definition of what we mean by $\mathbb{N}$. That turns out to involve some (usually) unspoken assumptions.

   b. In some developments, the principle of induction is taken as an explicit assumption about the natural numbers.

   c. An alternative is to make this assumption: Every subset of $\mathbb{N}$ has a first element. (As opposed say to the reals: the interval (0,1) does *not* have a first element.)

   d. With the first element assumption, we can prove the principle of math induction as follows: We argue by contradiction. Suppose that T is a set with properties 1 and 2, but that T is not equal to $\mathbb{N}$. Then there are elements of $\mathbb{N}$ that are not in T, and we define S to the set of all such elements. Then S is a subset of $\mathbb{N}$, so it has a first element z. In particular, z is not an element of T. This shows that z is not equal to 1 (which we know IS an element of T). Therefore $z > 1$, and consequently, $z - 1 \in \mathbb{N}$. But since z is the first element of S, $z - 1$ is NOT an element of S. That means $z - 1$ IS an element of T. But then by property 2, $(z - 1) + 1$ is also an element of T. That is, z is in T. This contradicts our earlier conclusion that z is not an element of T. Thus we have reached a contradiction, completing the proof.

6. Many examples of induction proofs are associated with integer identities, such as $1+2+ \cdots + n = n(n+1)/2$. But there are many other kinds of proofs that do not involve simple manipulation of equations. Let's look at a few examples.

   a. The determinant of an $n \times n$ upper triangular matrix is the product of the diagonal entries. Here we use the concept of an expansion by minors. This induction step is to relate a matrix of size n to one of size n+1.

For $n = 1$ we are considering the determinant of a 1×1. That has just one entry, which is by definition both the lone diagonal entry and the determinant. So the result trivially holds for $n = 1$. For $n = 2$ we can verify the result directly. The matrix must have the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, and the determinant is defined to be $ad - 0b = ad$. This is the product of the diagonal entries, so the result again holds.

Now assume the result holds for any upper diagonal matrix of size n and consider the following matrix of size $n + 1$: $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & a_{1\,n+1} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} & a_{2\,n+1} \\ 0 & 0 & a_{33} & \cdots & a_{3n} & a_{3\,n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} & a_{n\,n+1} \\ 0 & 0 & 0 & \cdots & 0 & a_{n+1\,n+1} \end{bmatrix}$. Expanding by minors in the first column, we get a_{11} times the determinant of the submatrix that remains after we cross out the first row and column. But that is an $n \times n$ upper triangular matrix, and so by our induction assumption, its determinant is the product $a_{22}\, a_{33} \cdots a_{n+1\,n+1}$. This shows that the determinant of the matrix shown above is $a_{11} \cdot a_{22}\, a_{33} \cdots a_{n+1\,n+1}$, which is the product of the diagonal entries. This shows that the result holds for an upper triangular matrix of size $n+1$, and completes the proof.

b. A set of n elements has 2^n subsets. Induction step: If $A = \{x_1, x_2, x_3, \cdots, x_n, x_{n+1}\}$ we partition subsets into two types: those that contain x_{n+1} and those that do not. Note that no set can be in both types, and every subset is one type or the other. By the induction hypothesis there are 2^n subsets of the first type. And there is a one-to-one correspondence between the two types. B is a subset of the first type if and only if $B \cup \{x_{n+1}\}$ is a subset of the second. Different B's produce different $B \cup \{x_{n+1}\}$'s, and every set of the second type is of the form $B \cup \{x_{n+1}\}$ for some B of the first type. This shows that there are 2^n subsets of the second type. Therefore, the total number of subsets is $2^n+2^n = 2\cdot 2^n = 2^{n+1}$.

c. For any natural number n, the product of n continuous functions is continuous. The proof of this uses the fact that the product of 2 continuous functions is continuous. For the induction step, we consider $n + 1$ continuous functions, denoted by $f_1, f_2, f_3, \cdots, f_n, f_{n+1}$. Let $g(x) = f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \cdots \cdot f_n(x)$. By the induction hypothesis, we know g is continuous. By the rule for a product of 2 continuous functions we also know that $g(x) \cdot f_{n+1}(x)$ is continuous. Therefore $f_1(x) \cdot f_2(x) \cdot f_3(x) \cdot \cdots \cdot f_n(x) \cdot f_{n+1}(x)$ is continuous.

7. Patterns peter out! This was discussed on day 1 with the example of cutting a pizza. See exercise 4.1.20. Induction proofs are an important in tool in verifying that a particular pattern does *NOT* peter out.

Section 4.2: Other forms of induction

1. Extended Principle of induction
   a. We don't always have to start with $n = 1$ as the starting point for induction (sometimes referred to as the *base case*. If we show $P(n) \rightarrow P(n+1)$, and that (say) $P(11)$ is true, then that shows $P(n)$ is true for all $n \ngeq 11$. Similarly, if we can show that $P(-6)$ is true, then we can conclude $P(n)$ is true for all $n \ngeq -6$.
   b. This is referred to in the text as the *Extended Principle of Math Induction*.
   c. Example from text: for $n \geq 4$, $n! > 2^n$.
      i. Direct calculation shows $4! > 2^4$ because $4! = 24$ and $2^4 = 16$
      ii. For induction step, assume that $n \geq 4$ and $n! > 2^n$. Then
      $$(n+1)! = (n+1)n! > (n+1)2^n \,. \qquad (1)$$
      From $n \geq 4$ we see that $n+1 > 2$. Therefore $(n+1)2^n > 2 \cdot 2^n = 2^{n+1}$. Combined with inequality (1), this shows that $(n+1)! > 2^{n+1}$. Thus the proposition holds for $n + 1$. This completes the induction step, and the proof.

2. Second Principle of Induction
   a. This is also sometimes called complete induction.
   b. For the induction step, we do not just assume that $P(n)$ holds for one particular n. Rather, we assume $P(k)$ holds for every k from the base case up to and including n, and show that $P(n+1)$ also holds.
   c. Example: Every natural number greater than 1 is either a prime or a product of primes.
      i. For this theorem we let $P(n)$ be "n is either a prime or a product of primes."
      ii. $P(2)$ is certainly true, because 2 is prime.
      iii. For the induction step, suppose that $P(k)$ is true for $2 \leq k \leq n$. We now verify $P(n+1)$. To proceed, we wish to show that $n+1$ is prime, or is a product of primes. Clearly, either $n + 1$ is prime or it isn't. If it *is* prime, there is nothing to show. So suppose $n + 1$ is not prime. That means there must be factors u and v greater than 1 such that $n + 1 = uv$. Now since each factor is greater than 1, we observe that each factor must also be less than $n + 1$. Thus, u and v are each between 2 and n, inclusive. Therefore, but the induction hypothesis, each is either prime, or is a product of primes. So when we multiply them, $n + 1 = uv$ is definitely a product of primes. Thus we have shown that either $n + 1$ is itself prime, or is a product of primes. This completes the induction and the proof.

End of Day