

Day 8: Friday, 2/10/2017

## 5.1 Sets

1. Overview: set notation and operations are everywhere in mathematics. Understanding this material and the basics of proving things with and about sets are crucial for further math study.
2. The basics: concept of a set as a collection. Roster, description, and set builder specification of sets. Finite and infinite sets, conceptually. The empty set. The universal set.
3. Set operations
  - a. Union
  - b. Intersection
  - c. Complement (relative to a universal set  $U$ ).  $A^c = \{x \in U \mid x \notin A\}$
  - d. Relative complement aka set subtraction or set difference:  $A - B$  is defined as the elements of  $A$  that are not also in  $B$ . Equivalently, these are elements that are in  $A$  but outside  $B$  (and hence in  $B^c$ ). Thus  $A - B = A \cap B^c$ .
4. Venn Diagrams
  - a. Graphical representation of set relationships
  - b. Like graphs in calculus, these diagrams can help you understand relationships, but are not considered adequate to prove results in most cases
  - c. There are diagrammatic interpretations for unions, intersections, complements, and combinations of these.
  - d. Example: Venn diagram shows that  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
5. Subsets and set equality
  - a.  $A = B$  iff they have identical elements. That means  $(\forall x \in U)((x \in A) \leftrightarrow (x \in B))$ .
  - b.  $A$  is a subset of  $B$  when every element of  $a$  is also an element of  $B$ . We also say that  $B$  contains  $A$  to mean the same thing as  $A$  is a subset of  $B$ .
  - c. Logically,  $A$  is a subset of  $B$  when  $(\forall x \in U)((x \in A) \rightarrow (x \in B))$ . Note that if  $A = B$  then also  $A$  is a subset of  $B$ . So *subset of* allows for the possibility of equality.
  - d. Notation is  $A \subseteq B$ , to remind us of the possibility of equality. We also write  $B \supseteq A$  ( $B$  contains  $A$ ) to mean the same thing as  $A \subseteq B$  ( $A$  is contained in  $B$ ).
  - e. Note that  $A = B$  iff  $A \subseteq B$  and  $B \subseteq A$ .
  - f. If  $A$  is a subset of  $B$  but is not equal to  $B$ , then  $A$  is contained in  $B$  but does not include every element of  $B$ . In this case we say that  $A$  is a proper subset of  $B$  and write  $A \subset B$ .
  - g. These notations are not universally accepted. In some works the symbol  $A \subset B$  means  $A$  is a subset of  $B$  (not necessarily proper) and  $A \subsetneq B$  means  $A$  is a proper subset of  $B$ .
6. Subset examples
  - a. Is  $\{x \in \mathbb{R} \mid 2x^3 + x^2 - 2x - 6 = 0\} \subseteq \mathbb{Q}$ ?
  - b. Is  $\{x \in \mathbb{R} \mid 2x^3 + x^2 - 2x - 6 = 0\} \subseteq \{x \in \mathbb{R} \mid (x^2 - 7)(2x^3 + x^2 - 2x - 6) = 0\}$ ?

- c. Is  $\{\text{continuous functions from } \mathbb{R} \text{ to } \mathbb{R}\} \subseteq \{\text{differentiable functions from } \mathbb{R} \text{ to } \mathbb{R}\}$ ? What if we replace  $\subseteq$  with  $\supseteq$ ? With  $\supset$ ?

## 7. Proving Subset and Equality Conditions (section 5.2)

- To prove  $A \subseteq B$  pick arbitrary element of  $A$  and prove it is also in  $B$
- Proof Skeleton: (see box at right)
- To prove  $\neg(A \subseteq B)$  (ie,  $A \not\subseteq B$ ), construct an element of  $A$  that is not in  $B$
- To use the assumption that  $A \subseteq B$ : Given that  $a \in A$ , conclude that  $a \in B$
- To use the assumption that  $\neg(A \subseteq B)$  (ie,  $A \not\subseteq B$ ): Say "Let  $a \in A$  such that  $a \notin B$ ." This is valid because the assumption implies that such an element exists.
- To prove  $A = B$  prove  $A \subseteq B$  and also prove  $B \subseteq A$ . This is like proving the two halves of an if and only if statement, and in fact what you are proving is  $x \in A$  iff  $x \in B$ .
- Example: Proposition 5.11. Let  $A$  and  $B$  are subsets of a universal set  $U$ . Then  $A - (A - B) = A \cap B$ . Prelim step, rewrite as  $A - (A \cap B^c) = A \cap B$ .

Proposition:  $A \subseteq B$ .  
 Proof: Let  $a \in A$ .  
 Then \_\_\_\_\_  
 so \_\_\_\_\_  
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 \_\_\_\_\_  
 so \_\_\_\_\_  
 so  $a \in B$ .

## 8. Disjoint Sets

- To prove a result of the form  $(P \rightarrow A \cap B = \emptyset)$ , it is often effective to argue by contradiction or using the contrapositive
- Contradiction: Assume  $P$  holds, and that there is an  $x$  is in both  $A$  and  $B$  and derive a contradiction.
- Contrapositive: Assume that there exists an  $x$  in both  $A$  and  $B$ , and prove that  $P$  is false.
- Example: Proposition 5.14. Let  $A$  and  $B$  are subsets of a universal set  $U$ . Then  $A \subseteq B$  iff  $A \cap B^c = \emptyset$ .

## 9. (Continuing 5.1) Empty set properties. For every set $B$ ...

- ...  $\emptyset \subseteq B \subseteq B$ .
- ...  $\emptyset \cup B = B$ .
- ...  $\emptyset \cap B = \emptyset$ .

## 10. Power Set of a Set

- Definition:  $\wp(A)$  is the set of subsets of  $A$ .
- Example:  $A = \{1, 2, 3\}$
- Don't confuse  $\in$  with  $\subseteq$ .  $\{1, 2\} \subseteq A$  but  $\{1, 2\} \in \wp(A)$
- Theorem: If  $A$  is a set with  $n$  elements, then  $A$  has  $2^n$  subsets (including  $\emptyset$  and  $A$  itself). In other words,  $\wp(A)$  has  $2^n$  elements.

## 11. Cardinality

- a. For a finite set this is the number of elements. We denote it  $\text{card}(A)$ . Some works write  $|A|$  for the cardinality of the set  $A$ .
  - b. In general, without knowing whether sets are finite or infinite, we define two sets to have equal cardinality iff there is a one to one correspondence between the two sets.
  - c. Set theoretic definition of infinite: The set  $A$  is infinite if and only if there is a proper subset  $B \subset A$  such that  $\text{card}(B) = \text{card}(A)$ .
  - d. Two infinite sets can have different cardinalities. For example  $\text{card}(\mathbb{R}) \neq \text{card}(\mathbb{Q})$ .
12. Section ends with review of some of the familiar standard number systems: natural numbers, integers, rationals, irrationals, reals, and complex numbers.

End of Day