Day 9: Tuesday, 2/14/2017
No quiz problem due Friday, exam a week from Friday.
Return hw. Collect hw regular hw 5.1 and first part of 5.2; one quiz problem. Take questions first.

Continuation of 5.2 material.

> Propositon: $A \subseteq B$.
> Proof: Let $a \in A$.
> Then ____
> so ______
>
> $\vdots$
>
> so ______
> so $a \in B$.

1. Proving Subset and Equality Conditions (section 5.2)
   a. To prove $A \subseteq B$ pick arbitrary element of A and prove it is also in B
   b. Proof Skeleton: (see box at right)
   c. To prove $\neg(A \subseteq B)$ (ie, $A \not\subseteq B$), construct an element of A that is not in B
   d. To use the assumption that $A \subseteq B$: Given that $a \in A$, conclude that $a \in B$
   e. To use the assumption that $\neg(A \subseteq B)$ (ie, $A \not\subseteq B$): Say "Let $a \in A$ such that $a \notin B$." This is valid because the assumption implies that such an element exists.
   f. To prove $A = B$ prove $A \subseteq B$ and also prove $B \subseteq A$. This is like proving the two halves of an if and only if statement, and in fact what you are proving is $x \in A$ iff $x \in B$.
   g. Example: Proposition 5.11. Let A and B are subsets of a universal set U. Then $A - (A - B) = A \cap B$. Prelim step, rewrite as $A - (A \cap B^c) = A \cap B$.
2. Disjoint Sets
   a. To prove a result of the form $(P \rightarrow A \cap B = \emptyset)$, it is often effective to argue by contradiction or using the contrapositive
   b. Contradiction: Assume P holds, and that there is an x is in both A and B and derive a contradiction.
   c. Contrapositive: Assume that there exists an x in both A and B, and prove that P is false.
   d. Example: Proposition 5.14. Let A and B are subsets of a universal set U. Then $A \subseteq B$ iff $A \cap B^c = \emptyset$.

3. (Continuing 5.1) Empty set properties. For every set B …
   a. … $\emptyset \subseteq B \subseteq B$.
   b. … $\emptyset \cup B = B$.
   c. … $\emptyset \cap B = \emptyset$.

4. Power Set of a Set
   a. Definition: $\wp(A)$ is the set of subsets of A.
   b. Example: $A = \{1, 2, 3\}$
   c. Don't confuse $\in$ with $\subseteq$. $\{1,2\} \subseteq A$ but $\{1,2\} \in \wp(A)$
   d. Theorem: If A is a set with n elements, then A has 2^n subsets (including $\emptyset$ and A itself). In other words, $\wp(A)$ has 2^n elements.

5. Cardinality
   a. For a finite set this is the number of elements. We denote it card(A). Some works write $|A|$ for the cardinality of the set A.
   b. In general, without knowing whether sets are finite or infinite, we define two sets to have equal cardinality iff there is a one to one correspondence between the two sets.
   c. Set theoretic definition of infinite: The set A is infinite if and only if there is a proper subset $B \subset A$ such that card(B) = card(A).
   d. Two infinite sets can have different cardinalities. For example card($\mathbb{R}$) $\neq$ card($\mathbb{Q}$).

Section ends with review of some of the familiar standard number systems: natural numbers, integers, rationals, irrationals, reals, and complex numbers.

New material, start 5.3 Properties of set operations

1. Algebraic Rules
   a. **Theorem 5.17.** *Let A, B, and C be subsets of some universal set U. Then*
      - $A \cap B \subseteq A$ *and* $A \subseteq A \cup B$.
      - *If* $A \subseteq B$, *then* $A \cap C \subseteq B \cap C$ *and* $A \cup C \subseteq B \cup C$.

   b. **Theorem 5.18 (Algebra of Set Operations).** *Let A, B, and C be subsets of some universal set U. Then all of the following equalities hold.*

Properties of the Empty Set and the Universal Set	$A \cap \emptyset = \emptyset$ $A \cup \emptyset = A$	$A \cap U = A$ $A \cup U = U$
Idempotent Laws	$A \cap A = A$	$A \cup A = A$
Commutative Laws	$A \cap B = B \cap A$	$A \cup B = B \cup A$
Associative Laws	$(A \cap B) \cap C = A \cap (B \cap C)$ $(A \cup B) \cup C = A \cup (B \cup C)$	
Distributive Laws	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	

   c. These are all equalities of sets and can be proved by showing two subset results – each set is shown to be a subset of the other. The book also shows an argument where each statement is an iff statement, and together they form an iff chain between two statements: $x \in A$ and $x \in B$. This shows $x \in A$ iff $x \in B$ so $A = B$.

   d. Sample proof: second distributive law.

2. Properties of the Complement Operation

   a. **Theorem 5.20.** *Let A and B be subsets of some universal set U. Then the following are true:*

Basic Properties	$(A^c)^c = A$ $A - B = A \cap B^c$
Empty Set and Universal Set	$A - \emptyset = A$ *and* $A - U = \emptyset$ $\emptyset^c = U$ *and* $U^c = \emptyset$
De Morgan's Laws	$(A \cap B)^c = A^c \cup B^c$ $(A \cup B)^c = A^c \cap B^c$
Subsets and Complements	$A \subseteq B$ *if and only if* $B^c \subseteq A^c$

   b. Sample proof: Second De Morgan law

3. Proving collections of statements are equivalent
   a. Overview: common practice when we have three or more different equivalent characterizations of something. Example: Invertible Matrix Theorem.
   b. A common strategy is to prove $P_1 \to P_2$ and $P_2 \to P_3$ and $P_3 \to P_4$ and $P_4 \to P_1$ (which we abbreviate by writing $P_1 \to P_2 \to P_3 \to P_4 \to P_1$). This also proves that any two of the statements are equivalent. For example, we can show that $P_2 \leftrightarrow P_4$ as follows. By transitivity, $P_2 \to P_3$ and $P_3 \to P_4$ implies $P_2 \to P_4$. Conversely, since $P_4 \to P_1$ and $P_1 \to P_2$, we conclude $P_4 \to P_2$. Thus we have shown $P_2 \leftrightarrow P_4$.
   c. This idea is illustrated in the proof of Theorem 5.22.

Theorem 5.22. *Let A and B be subsets of some universal set U. The following are equivalent:*

1. $A \subseteq B$ ***2.*** $A \cap B^c = \emptyset$ ***3.*** $A^c \cup B = U$

(for 1 → 2 use contrapositive; for 3 → 1 chase an element of *A* and show it is in *B*.)

End of Day