An Alternate Definition of the Function Concept

Our text defines the concept of a function in terms of a rule or procedure. But this is a bit vague for careful mathematical analysis – different people might have different ideas about what constitutes a rule. The more modern view point is to define a function in terms of a special kind of set. More specifically, a function from A to B is defined in terms of a subset of the cartesian product $A \times B$.

> Following this page is a short excerpt from *Proofs and Concepts: the fundamentals of abstract mathematics*, by Dave Witte Morris and Joy Morris, University of Lethbridge.
>
> Available on line at https://open.umn.edu/opentextbooks/BookDetail.aspx?bookId=395.

According to the definition in our text, when we write $f : A \to B$, that means for every $a \in A$, there is associated a unique element of B, denoted $f(a)$. Accordingly, we can define the set

$$G(f) = \{(a, f(a)) \mid a \in A\} \subseteq A \times B.$$

In many presentations, this set is referred to as the *Graph* of the function f. Indeed, in the familiar setting of calculus, when $f : \mathbb{R} \to \mathbb{R}$, the geometric graph of f is a visual portrayal of exactly the set of ordered pairs defined as $G(f)$ above.

In the following excerpt, a different approach is taken. As you will see, the excerpt defines the concept of *function from A to B* to be a certain kind of subset of $A \times B$, namely, the very sort of subset defined above. Thus, instead of the graph of the function being a set that is defined using the function, the graph of the function *IS* the function. This is a more logically rigorous way to define the function concept, because it avoids the ambiguous term *rule*, and uses instead familiar concepts from set theory. And this alternative definition is the one that is most commonly accepted in advanced mathematics.

As part of your preparation for more advanced mathematics classes, it is worthwhile for you to be familiar with both versions of the definition of *function*.

In the excerpt, section 6.2 gives an informal explanation of how a function can be understood to be a certain set of ordered pairs. Then section 6.3 gives a formal definition of the concept of function in terms of sets of ordered pairs.

6.2. Informal introduction to functions

You have seen many examples of functions in your previous math classes. Most of these were probably given by formulas (such as $f(x) = x^3$), but functions can also be given in other ways. The key property of a function is that it accepts inputs, and provides a corresponding output value for each possible input.

EXAMPLE 6.2.1. For the function $f(x) = x^3$, the input x can be any real number. Plugging a value for x into the formula yields an output value, which is also a real number. For example, using $x = 2$ as the input yields the output value $f(2) = 2^3 = 8$.

DEFINITION 6.2.2 (unofficial). Suppose f is any function.

1) The set of allowable inputs of f is called the **domain** of f.
2) If A is the domain of f, and B is any set that contains all of the possible outputs of f, then we say that f is a **function from A to B**. In the case of the function $f(x) = x^3$, we may take A and B to both be the set of real numbers; thus, f is a function from $\mathbb{R}$ to $\mathbb{R}$.

EXAMPLE 6.2.3. $g(x) = 1/x$ is *not* a function from $\mathbb{R}$ to $\mathbb{R}$. This is because 0 is an element of $\mathbb{R}$, but the formula does not define a value for $g(0)$. Thus, 0 cannot be in the domain of g. To correct this problem, one could say that g is a function from the set $\{\, x \in \mathbb{R} \mid x \neq 0 \,\}$ of *nonzero* real numbers, to $\mathbb{R}$.

Intuitively, a function from A to B can be thought of being any process that accepts inputs from the set A, and assigns an element of the set B to each of these inputs. The process need not be given by a formula. Indeed, most of the functions that arise in science or in everyday life are not given by any formula.

EXAMPLE 6.2.4.

1) Each point on the surface of the earth has a particular temperature right now, and the temperature (in degrees centigrade) is a real number. Thus, temperature defines a function temp from the surface of the earth to $\mathbb{R}$: temp(x) is the temperature at the point x.
2) The items in a grocery store each have a particular price, which is a certain number of cents, so price can be thought of as a function from the set of items for sale to the set $\mathbb{N}$ of all natural numbers: price(x) is the price of item x (in cents).
3) If we let People be the set of all people (alive or dead), then mother is a function from People to People. For example,

   mother(Prince Charles) = Queen Elizabeth.

   (To avoid ambiguity, perhaps we should clarify that, by "mother," we mean "biological mother.")
4) In contrast, grandmother is *not* a function from People to People. This is because people have not just one grandmother, but two (a maternal grandmother and a paternal grandmother). For example, if we say that Prince Charles wrote a poem for his grandmother, we do not know whether he wrote the poem for the mother of Queen Elizabeth, or for his other grandmother. A function is not ever allowed to have such an ambiguity. (In technical terms, grandmother is a "relation," not a function. This will be explained in Section 7.1.)

Functions are often given by a *table* of values.

EXAMPLE 6.2.5. The list of prices in a store is an example of this:

item	price (in cents)
apple	65
banana	83
cherry	7
donut	99
egg	155

In this example:

- The domain of price is $\{\text{apple}, \text{banana}, \text{cherry}, \text{donut}, \text{egg}\}$.
- $\text{price}(\text{banana}) = 83$.
- price(guava) does not exist, because guava is not in the domain of the function.

Instead of making a table, mathematicians prefer to represent each row of the table by an ordered pair. For example, the first row of the table is apple | 65. This has apple on the left and 65 on the right, so we represent it by the ordered pair (apple, 65), which has apple on the left and 65 on the right. The second row is represented by (banana, 83). Continuing in this way yields a total of 5 ordered pairs (one for each row). To keep them gathered together, a mathematician puts them into a set. Thus, instead of writing a table, a mathematician would represent this function as:

$$\{\, (\text{apple}, 65), (\text{banana}, 83), (\text{cherry}, 7), (\text{donut}, 99), (\text{egg}, 155) \,\}.$$

The set of ordered pairs contains exactly the same information as a table of values, but the set is a more convenient form for mathematical manipulations.

EXERCISE 6.2.6. Here is a function f given by a table of values.

x	$f(x)$
1	7
2	3
3	2
4	4
5	9

1) What is the domain of f?
2) What is $f(3)$?
3) Represent f as a set of ordered pairs.
4) Find a formula to represent f.
   [*Hint:* There is a formula of the form $f(x) = ax^2 + bx + c$.]

EXAMPLE 6.2.7. Not every table of values represents a function. For example, suppose we have the following price list, which is a slight change from Example 6.2.5:

item	price (in cents)
apple	65
banana	83
cherry	7
donut	99
banana	155

There is a problem here, because there are two possible prices for a banana, depending on which line of the table is looked at. (So you might pick up a banana, expecting to pay 83 cents, and end up having the cashier charge you $1.55.) This is not allowed in a function: each input must have exactly one output, not a number of different possible outputs. Thus, if a table represents a function, and an item appears in the left side of more than one row, then all of those rows must have the same output listed on the right side.

Remark 6.2.8. A 2-column table represents a function from A to B if and only if:

1) every value that appears in the left column of the table is an element of A,
2) every value that appears in the right column of the table is an element of B,
3) every element of A appears in the left side of the table, and
4) no two rows of the table have the same left side, but different right sides.

EXAMPLE 6.2.9. Which of the following are functions from $\{1, 2, 3\}$ to $\{\mathsf{w}, \mathsf{h}, \mathsf{o}\}$? (If it is not such a function, then explain why not.)

1) $\{(1, \mathsf{w}), (1, \mathsf{h}), (1, \mathsf{o})\}$
2) $\{(1, \mathsf{h}), (2, \mathsf{h}), (3, \mathsf{h})\}$
3) $\{(1, \mathsf{h}), (2, \mathsf{o}), (3, \mathsf{w})\}$
4) $\{(\mathsf{w}, 1), (\mathsf{h}, 2), (\mathsf{o}, 3)\}$

SOLUTION.

(1) This is not a function. Since $(1, \mathsf{w})$, $(1, \mathsf{h})$, and $(1, \mathsf{o})$ are all in the set, there are three different elements b (not a *unique* b), such that $(1, b)$ is in the set.

(2) This is such a function.

(3) This is such a function.

(4) This is not such a function, because, for the element $(\mathsf{w}, 1)$ of the set, there do not exist elements a of $\{1, 2, 3\}$ and b of $\{\mathsf{w}, \mathsf{h}, \mathsf{o}\}$, such that $(\mathsf{w}, 1) = (a, b)$. (Instead, we would need to take a in $\{\mathsf{w}, \mathsf{h}, \mathsf{o}\}$ and b in $\{1, 2, 3\}$, which is backwards from what is required. In fact, f is a function from $\{\mathsf{w}, \mathsf{h}, \mathsf{o}\}$ to $\{1, 2, 3\}$, not from $\{1, 2, 3\}$ to $\{\mathsf{w}, \mathsf{h}, \mathsf{o}\}$.) □

EXERCISE 6.2.10. Let

- $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$, and
- $B = \{1, 3, 5, 7, 9, 11\}$.

Which of the following sets of ordered pairs are functions from A to B? (For those that are not, explain why.)

1) $\{(\mathsf{a}, 1), (\mathsf{b}, 3), (\mathsf{c}, 5), (\mathsf{d}, 7), (\mathsf{e}, 9)\}$
2) $\{(\mathsf{a}, 1), (\mathsf{b}, 2), (\mathsf{c}, 3), (\mathsf{d}, 4), (\mathsf{e}, 5)\}$
3) $\{(\mathsf{a}, 1), (\mathsf{b}, 3), (\mathsf{c}, 5), (\mathsf{d}, 3), (\mathsf{e}, 1)\}$
4) $\{(\mathsf{a}, 1), (\mathsf{b}, 3), (\mathsf{c}, 5), (\mathsf{d}, 7), (\mathsf{e}, 9), (\mathsf{a}, 11)\}$
5) $\{(\mathsf{a}, 1), (\mathsf{b}, 3), (\mathsf{c}, 5), (\mathsf{e}, 7)\}$
6) $\{(\mathsf{a}, 1), (\mathsf{b}, 1), (\mathsf{c}, 1), (\mathsf{d}, 1), (\mathsf{e}, 1)\}$
7) $\{(\mathsf{a}, \mathsf{a}), (\mathsf{b}, \mathsf{a}), (\mathsf{c}, \mathsf{a}), (\mathsf{d}, \mathsf{a}), (\mathsf{e}, \mathsf{a})\}$
8) $\{(\mathsf{a}, 1), (\mathsf{b}, 3), (\mathsf{c}, 5), (\mathsf{d}, 5), (\mathsf{e}, 3), (\mathsf{a}, 1)\}$
9) $\{(1, \mathsf{a}), (3, \mathsf{a}), (5, \mathsf{a}), (7, \mathsf{a}), (9, \mathsf{a}), (11, \mathsf{a})\}$
10) $\{(\mathsf{c}, 1), (\mathsf{b}, 3), (\mathsf{e}, 5), (\mathsf{a}, 7), (\mathsf{d}, 9)\}$

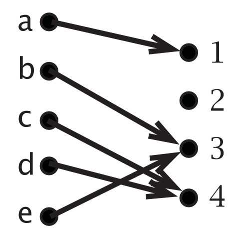
Remark 6.2.11. It is sometimes helpful to represent a function $f\colon A \to B$ by drawing an **arrow diagram**:

- a dot is drawn for each element of A and each element of B, and
- an arrow is drawn from a to $f(a)$, for each $a \in A$.

For example, suppose

- $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e}\}$,
- $B = \{1, 2, 3, 4\}$, and
- $f = \{(\mathsf{a}, 1), (\mathsf{b}, 3), (\mathsf{c}, 4), (\mathsf{d}, 4), (\mathsf{e}, 3)\}$.

Then the picture at right is an arrow diagram of f.



Notice that:

1) There is exactly one arrow coming out of each element of A. This is true for the arrow diagram of any function.
2) There can be any number of arrows coming into each element of B (perhaps none, perhaps one, or perhaps more than one).

6.3. Official definition

The preceding section provided some intuition about how and why functions are represented as sets of ordered pairs, but it is not at all authoritative. Here are the official definitions.

DEFINITION 6.3.1. Suppose A and B are sets.

1) A set f is a **function from A to B** iff
   (a) each element of f is an ordered pair (a, b), such that $a \in A$ and $b \in B$, and
   (b) for each $a \in A$, there is a *unique* $b \in B$, such that $(a, b) \in f$.
2) If f is a function from A to B, then
   - A is called the **domain** of f, and
   - B is a **codomain** of f.
3) We write "$f \colon A \to B$" to denote that f is a function from A to B.

EXERCISE 6.3.2. We can express the definition of a function in First-Order Logic:

1) Translate the assertion of Definition 6.3.1(1a) into First-Order Logic.
2) Translate the assertion of Definition 6.3.1(1b) into First-Order Logic.

NOTATION 6.3.3. Suppose $f \colon A \to B$.

1) For $a \in A$, it is convenient to have a name for the element b of B, such that $(a, b) \in f$. The name we use is $f(a)$:

   $f(a) = b$ if and only if $(a, b) \in f$.

2) Each element a of A provides us with an element $f(a)$ of B. The **range** of f is the set that collects together all of these elements $f(a)$. That is,

   b is in the range of f iff there is some $a \in A$, such that $b = f(a)$.

   The range can be denoted $\{\, f(a) \mid a \in A \,\}$.

EXAMPLE 6.3.4. Suppose the function f is defined by $f(x) = x^2$, on the domain $\{0, 1, 2, 4\}$. Then:

1) To represent f as a set of ordered pairs, each element of the domain must appear exactly once as a first coordinate, with the corresponding output given in the second coordinate. Since there are four elements in the domain, there will be four ordered pairs: $f = \{(0, 0), (1, 1), (2, 4), (4, 16)\}$.

2) To give a table for f, we include one row for every element of the domain. The table will be:

n	$f(n)$
0	0
1	1
2	4
4	16

3) If we are asked what is $f(3)$, the answer is that $f(3)$ *does not exist*, because 3 is not in the domain of f. Even though we know that $3^2 = 9$, the formula we gave for f only applies to elements that are in the domain of f! It is not true that $f(3) = 9$.
4) The range of f is the set of possible outputs: in this case, the range of f is $\{0, 1, 4, 16\}$.
5) If we are asked what is $f(2)$, the answer is $f(2) = 4$.
6) Is f a function from $\{n \in \mathbb{N} \mid n \leq 4\}$ to $\{0, 1, 4, 16\}$? The answer is no, because the first set is $\{0, 1, 2, 3, 4\}$, which includes the value 3, but 3 is not in the domain of f.
7) Is f a function from $\{0, 1, 2, 4\}$ to $\{n \in \mathbb{N} \mid n \leq 16\}$? The answer is yes; even though the second set has many values that are not in the range, it is a possible codomain for f. A codomain can be any set that contains all of the elements of the range, so every function has many different codomains (but only one domain and only one range).

EXERCISES 6.3.5.

1) The table at right describes a certain function g.

n	$g(n)$
2	7
4	9
6	11
8	13
10	15

   (a) What is the domain of g?
   (b) What is the range of g?
   (c) What is $g(6)$?
   (d) What is $g(7)$?
   (e) Represent g as a set of ordered pairs.
   (f) Draw an arrow diagram to represent g.
   (g) Write down a formula that describes g.
   (Express $g(n)$ in terms of n, by using simple arithmetic operations.)

2) Suppose
   - f is a function whose domain is $\{0, 2, 4, 6\}$, and
   - $f(x) = 4x - 5$, for every x in the domain.

   Describe the function in each of the following ways:
   (a) Make a table. (b) Draw an arrow diagram. (c) Use ordered pairs.

3) For the given sets A and B:
   (i) Write each function from A to B as a set of ordered pairs.
   (ii) Write down the range of each function.

   (a) $A = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$, $B = \{\mathsf{d}\}$
   (b) $A = \{\mathsf{a}, \mathsf{b}\}$, $B = \{\mathsf{c}, \mathsf{d}\}$
   (c) $A = \{\mathsf{a}\}$, $B = \{\mathsf{b}, \mathsf{c}, \mathsf{d}\}$
   (d) $A = \{\mathsf{a}, \mathsf{b}\}$, $B = \{\mathsf{c}, \mathsf{d}, \mathsf{e}\}$

   [*Hint:* For (i), you may assume, without proof, that if A has exactly m elements, and B has exactly n elements, then the number of functions from A to B is n^m. (Do you see why this is the correct number?)]