

Recursively Defined Sequences

Our text provides a very general formulation of the concept of a recursively defined sequence. In this handout we will consider a special subset of such sequences. The basic idea is to start with an initial value, and then apply a specific function repeatedly.

Example 1: Let $f(x) = \frac{1}{1+x}$. If we start with 1 and apply f repeatedly we generate the following results

$$1 \xrightarrow{f} \frac{1}{2} \xrightarrow{f} \frac{1}{1+\frac{1}{2}} = \frac{2}{3} \xrightarrow{f} \frac{1}{1+\frac{2}{3}} = \frac{3}{5} \text{ etc}$$

This sequence can be extended to as many terms as we wish recursively.

From calculus 2, we know that one of the most basic questions about a sequence is whether it converges, and if so, what the limit is. We will see in this handout that these questions can often be answered for recursively defined sequences, and that induction plays a major role.

In developing these ideas, we will use several results from calculus.

Bounded Monotone Sequence Theorems. Suppose a_1, a_2, a_3, \dots is a sequence with these properties:

1. $a_1 \leq a_2 \leq a_3 \leq \dots$ (we say this sequence is increasing)
2. There exists a constant C such that $a_n \leq C$ for every n (we say this sequence is bounded above)

Then the sequence converges to a finite limit L with $a_1 \leq L \leq C$.

Similarly, if the sequence is decreasing and bounded below the sequence converges to a finite limit.

Monotone Function Theorem. Suppose that $f(x)$ is differentiable and $f'(x) \geq 0$ for all x in some interval. Then if x_1 and x_2 are elements of the interval, and if $x_1 \leq x_2$ then $f(x_1) \leq f(x_2)$. (We say that the function f is increasing.)

Continuous Functions and Limits of Sequences. If the sequence a_1, a_2, a_3, \dots converges to a finite limit L , and if the function $f(x)$ is continuous at $x = L$, then the sequence $f(x_1), f(x_2), f(x_3), \dots$ also converges with limit $f(L)$. In symbols, $\lim_{n \rightarrow \infty} f(x_n) = f(\lim_{n \rightarrow \infty} x_n)$.

We now look at a specific example to illustrate how these results can be used.

Example 2. Let $f(x) = 1 + \sqrt{x}$. Define a sequence $a_0, a_1, a_2, a_3, \dots$ as follows: $a_0 = 1$, and for each natural number n , $a_n = f(a_{n-1})$. Thus $a_1 = 1 + \sqrt{a_0} = 1 + \sqrt{1} = 2$, $a_2 = 1 + \sqrt{a_1} = 1 + \sqrt{2}$, $a_3 = 1 + \sqrt{a_2} = 1 + \sqrt{1 + \sqrt{2}}$, and so forth. We will show that this sequence is increasing and bounded, and hence has a limit L . Then we will find an exact expression for L .

Bounded Above. We will show that $a_n < 9$ for every n , using induction. To begin, notice that by definition $a_0 = 1$ is less than 9. Also we saw above that $a_1 = 2$, and that is also less than 9. Now suppose that n is a specific natural number with $a_n < 9$. Then $a_{n+1} = f(a_n) = 1 + \sqrt{a_n} < 1 + \sqrt{9} = 1 + 3 = 4 < 9$. Therefore, by induction $a_n < 9$ for every $n \geq 0$.

Bounded Below. We next show that each a_n is positive. As before, we argue by induction. The values of a_0 and a_1 are 1 and 2, both of which are positive. For the induction step, suppose that $a_n > 0$. Then $f(a_n) = 1 + \sqrt{a_n}$ is defined and we have $a_{n+1} = f(a_n) = 1 + \sqrt{a_n} > 1$, and that implies $a_{n+1} > 0$. Therefore, by induction, $a_n > 0$ for all $n \geq 0$.

Increasing. For $x > 0$, the function $f(x)$ is differentiable, and $f'(x) = \frac{1}{2\sqrt{x}} > 0$. This shows that f is an increasing function for $x > 0$. Now we use this to prove each a_n is less than the following term a_{n+1} . Arguing by induction, observe that $a_0 \leq a_1$. For the induction step, suppose that $n \geq 0$ is an integer such that $a_n \leq a_{n+1}$. We have already shown that both a_n and a_{n+1} are positive, and so we may apply f to each. Moreover, since f is an increasing function, we must have $f(a_n) \leq f(a_{n+1})$. But by definition $f(a_n) = a_{n+1}$ and $f(a_{n+1}) = a_{n+2}$. Thus we have $a_{n+1} \leq a_{n+2}$. That is, we have shown that when $a_n \leq a_{n+1}$ holds so does $a_{n+1} \leq a_{n+2}$. Thus, by induction, $a_n \leq a_{n+1}$ for every n . That is, the sequence is increasing.¹

Existence of the Limit. We have now shown that the sequence is increasing and bounded above. By the Bounded Monotone Sequence Theorem, there must be a finite limit L

Finding the Limit. We know

$$\lim_{n \rightarrow \infty} a_n = L.$$

We also know that $f(x)$ is a continuous function for all positive x . Therefore, by the theorem on continuous functions and limits of sequences, we can write

$$\lim_{n \rightarrow \infty} f(a_n) = f(L).$$

On the other hand, we know that for every n , $f(a_n) = a_{n+1}$. Substituting shows

$$\lim_{n \rightarrow \infty} a_{n+1} = f(L). \quad (1)$$

But what is meant by the limit of a_{n+1} ? With $n = 0, 1, 2, 3$, etc, we see that $n+1$ takes on the values 1, 2, 3, etc. That is, we are looking at the limit of a_1, a_2, a_3, \dots and this is clearly the same as the limit of $a_0, a_1, a_2, a_3, \dots$, which we know to be L . In symbols, $\lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} a_n = L$. Substituting in equation (1) thus gives the equation $L = f(L)$.

Using the definition of f now produces $L = 1 + \sqrt{L}$. We subtract 1 from both sides and square to obtain $(L - 1)^2 = L$. This can be rewritten in the form $L^2 - 2L + 1 = L$, so $L^2 - 3L + 1 = 0$. In this we have shown that the limit L must be a root of the preceding quadratic equation.

Now we can use the quadratic formula to find the roots as $\frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$. These are given approximately by a calculator as 2.618 and 0.382. We know that L must be one of these numbers. But we also know that the sequence is increasing and each successive term is further above $a_0 = 1$. Therefore the limit cannot be less than 1. We conclude that the limit must be the larger root, namely $\frac{3 + \sqrt{5}}{2}$.

An Infinite Calculation. In symbolic form, we can write the first few terms of our sequence as

$$1, \quad 1 + \sqrt{1}, \quad 1 + \sqrt{1 + \sqrt{1}}, \quad 1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}, \quad 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1}}}}, \dots$$

¹ For a sequence given by $a_{n+1} = f(a_n)$, just knowing that f is an increasing function does not imply that a_1, a_2, a_3, \dots is an increasing sequence. As an example, let $f(x) = 1 - 1/x$ and $a_1 = 2$. You can easily show that f is an increasing function. Computing the first several terms of the sequence will show that these terms are not increasing.

It is natural to consider the limit of these terms to be given by an infinite calculation of the form

$$1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$$

which we interpret as involving the nesting of an infinite number of square roots. If we represent this value by L , we can show that $1 + \sqrt{L} = L$ as follows. Begin with

$$L = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}$$

and take the squareroot of both sides. That produces

$$\sqrt{L} = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}} .$$

Adding 1 to both sides now gives

$$1 + \sqrt{L} = 1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \dots}}}}} .$$

But notice that the infinite calculation on the right is exactly the one we have defined as L . Thus we have derived the equation $L = 1 + \sqrt{L}$ which we found above. However, in the earlier analysis we had a rigorous basis for deducing that the limit L exists and must satisfy the equation $L = 1 + \sqrt{L}$. In contrast, while the manipulation of symbols representing an infinite calculation seems very natural (and in this case leads to a correct answer), it is not of itself rigorous. The combination of the suggestive use of symbolic manipulation and the careful derivation using logical proofs constitutes a powerful tool of mathematics.

Some exercises are shown on the next page.

Exercises

4.3.x1 Follow the outline of the example above to analyze this infinite calculation:

$$2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}$$

To do so, consider the function $f(x) = 2 + \sqrt{x}$ and define a sequence using the rule $a_n = f(a_{n-1})$. Prove that the sequence is bounded and increasing, and therefore has a limit L . Then show that L has to be a root of a specific equation, and proceed to determine what that root is.

4.3.x2 Use a similar approach with the function $f(x) = \sqrt{2x}$ defining $a_0 = 1$, and for each natural number n , $a_n = f(a_{n-1})$. Show that this sequence has a limit, and determine what that limit is. Then express the limit in the form of an infinite calculation of the sort mentioned in the preceding exercise.

4.3.x3 Use the methods of this handout to analyze this infinite calculation:

$$3 - \frac{1}{3 - \frac{1}{3 - \frac{1}{3 - \dots}}}$$

Hint: Use the function : $f(x) = 3 - \frac{1}{x}$.

4.3.x4 If f is as in the preceding problem, $a_0 = 1$, and for each natural number n , $a_n = f(a_{n-1})$, show by induction that

$$a_n = 3 - \frac{F_{2n-3}}{F_{2n-1}}$$

where F_n is the n^{th} Fibonacci number. Then, using your value for $L = \lim_{n \rightarrow \infty} a_n$, find an exact expression for the limiting value of the ratio of successive odd terms of the Fibonacci sequence. That is, find an exact expression for $\lim_{n \rightarrow \infty} \frac{F_{2n-3}}{F_{2n-1}}$.