Analyzing Fibonacci Numbers with Difference Equations

Overview

In class we discussed difference equations and progressions or sequences of vectors. The general idea is this. Start with a fixed matrix A and a given vector $\mathbf{v}_0$. Then apply A repeatedly to generate a sequence of vectors:

$$\mathbf{v}_1 = A\,\mathbf{v}_0$$
$$\mathbf{v}_2 = A\,\mathbf{v}_1$$
$$\mathbf{v}_3 = A\,\mathbf{v}_2$$
$$\vdots$$

In this handout you will see that the Fibonacci numbers can be generated using a difference equation approach. To review, the Fibonacci numbers make up the sequence 1, 1, 2, 3, 5, 8, 13, … in which each term is the sum of the two preceding terms. The first one is considered the first Fibonacci number. Although not shown, we can also include the zero-th Fibonacci number, 0.

Vectors of Fibonacci Numbers.

For our difference equation analysis, we consider the vectors

$$\mathbf{v}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$$

where F_n is the nth Fibonacci number. For example,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},\quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},\quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ and } \mathbf{v}_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

We also have to understand the process for going from one vector in the sequence to the next. That can be described by a matrix as follows:

$$\mathbf{v}_{n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{v}_n .$$

This works because

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n + F_{n+1} \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix}$$

The *n* th Fibonacci Number

Using the difference equation set up presented on the preceding page, we can see that

$$\mathbf{v}_n = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \mathbf{v}_0$$

or in more detail

$$\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This gives a computational algorithm for the *n*th Fibonacci number: multiply out the right-hand side of the preceding equation, and then take the top entry of the resulting vector. Alternatively, to obtain the top entry of the vector, we can take its dot product with $[1\ \ 0]^T$. That leads to

$$F_n = [1\ 0] \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Using the eigenvalues and eigenvectors of the matrix in this equation, we can actually come up with a direct algebraic expression for the *n*th Fibonacci number. An outline for this is shown next. This same outline appears in the homework for section 5.2.

Problems.

Let the matrix A be given by $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

1. Find the eigenvalues of A

2. For each eigenvalue, find one eigenvector. Call these eigenvectors **u** and **v**.

3. Express $[1\ \ 1]^T$ as a linear combination of **u** and **v**.

4. Find a formula for $A^k[1\ \ 1]^T$ using the same methods as in the extra problems for section 5.1. Multiply by $[1\ \ 0]^T$ to obtain a formula for the *n*th Fibonacci number.

5. By direct calculation with $k = 6$, verify that your formula produces the 6th and 7th Fibonacci numbers.