Linear Algebra: Sample Questions for Exam 2

Instructions: This is not a comprehensive review: there are concepts you need to know that are not included. Be sure you study all the sections of the book and the handouts we covered. You should know correct statements of definitions and important theorems, be able to state concepts in your own words, and correctly apply procedures. For full credit on the real exam, you will have to show work or give some explanation of how you reached each answer. That is, you will be expected to communicate to me how you reached your answer. You should practice doing that in answering these sample questions.

Note: Some of the following pages are from exams I have used in prior semesters. As a result, the page and problem numbering is inconsistent. Also, in some problems the set of real numbers is represented by a regular boldface **R,** rather than the symbol $\mathbb{R}$ from class handouts and the text.

1. Subspaces
   A. What is the definition of a subspace of a vector space? **Answer:** By definition, a subspace is a subset that is a vector space in its own right. In practice, this means that it is a nonempty subset that is closed under addition and scalar multiplication.

   B. In your own words, explain the concept of subspace. **Answer:** Conceptually, a subspace is a subset that is a closed system with respect to the operations of linear algebra. That is, linear combinations of the elements of the set always remain in the set. Accordingly, we can formulate vector equations involving the elements of the set and the solutions will always be found within the same set.

   C. Give an example of a subspace of $\mathbb{R}^3$. How do you know that your example IS a subspace? **Answer:** Two trivial examples are $\{\mathbf{0}\}$, the set just containing the zero vector, and the entire space $\mathbb{R}^3$. A more interesting example is the column space of the matrix $\begin{bmatrix} 1 & 3 & 2 & 5 \\ 0 & 4 & 2 & 6 \\ -2 & 6 & 2 & 8 \end{bmatrix}$. We know this is a subspace of $\mathbb{R}^3$ because we saw a theorem that said the column space of an $m \times n$ matrix is always a subspace of $\mathbb{R}^m$.

   D. Give an example of a subset of $\mathbb{R}^3$ that is NOT a subspace. How do you know that your example is NOT a subspace. **Answer:** No finite set of nonzero vectors can be a subspace. This is because a subspace has to be closed under scalar multiplication. Since there are infinitely many scalars, a subspace has to contain the infinitely many scalar multiples of any nonzero vector in the subspace. A finite set cannot contain all those multiples, and so is not closed under scalar multiplication, and hence not a subspace.

4. Definitions State the definitions of the items on this and the next page.

a. The column space of a matrix

Answer: The column space of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ spanned by the columns of A. That is, it is the set of all linear combinations of the columns of A.

b. The null space of a matrix.

Answer: The null space of an $m \times n$ matrix A is the set of solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. This is a subspace of $\mathbb{R}^n$, and consists of all the vectors in $\mathbb{R}^n$ that are perpendicular to all the rows of A.

c. A linearly independent set of vectors in a vector space V.

Answer: The set $\{v_1, v_2, \ldots, v_p\}$ is linearly independent if the equation $c_1v_1 + c_2v_2 + \ldots + c_pv_p = 0$ holds if and only if all the scalars c_j are zero.

d. The dimension of a vector space V.

Answer: If V can be spanned by a finite set, then the dimension is the number of elements in any basis. That is, the dimension is n if and only if any linearly independent spanning set has n elements. If V cannot be spanned by any finite set V is infinite dimensional.

e. Column rank, row rank, and rank of an $m \times n$ matrix A.

Answer: The column rank is the dimension of the column space, and equal to the maximum number of independent columns. It is also equal to the number of pivot columns in A. The row rank is the dimension of the row space and equal to the maximum number of independent rows. It is also equal to the number of nonzero rows in the rref of A. We say a theorem that the row and column rank of a matrix always equal each other (because the number of pivot columns equals the number of pivot entries, and that is equal to the number of nonzero rows in the rref.) The rank of the matrix is the common value of the row and column ranks. That is, row rank = column rank = rank.

5. Cramer's Rule

a. State Cramer's rule for solving a linear system $A\mathbf{x} = \mathbf{b}$ where A is an invertible $n \times n$ matrix, $\mathbf{b}$ is a given vector in $\mathbf{R}^n$, and $\mathbf{x}$ is an unknown vector in $\mathbf{R}^n$.

Let A be an $n \times n$ invertible matrix, and let b be a vector in $\mathbb{R}^n$. For each j between 1 and n inclusive, define a matrix B_j by replacing the j th column of A by b. Then Cramer's rule states that the unique solution to the system $Ax = b$ is the vector $x=[\, x_1\ x_2 \ldots x_n]^T$ where $x_j = \det(B_j)/\det(A)$ for all j.

b. Use Cramer's rule to solve the system

$$\begin{bmatrix} 3 & 7 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

Let A be the matrix on the left side of the equation. We can see that $\det(A) = (3)(4) – (2)(7) = -2$. Now replace the first column of A with $[5\ 11]^T$ to make B_1 and replace the second column with $[5\ 11]^T$ to make B_2 and take the determinants. We find

$$\det B_1 = \begin{vmatrix} 5 & 7 \\ 11 & 4 \end{vmatrix} = 20 - 77 = -57$$

and

$$\det B_2 = \begin{vmatrix} 3 & 5 \\ 2 & 11 \end{vmatrix} = 33 - 10 = 23 \cdot$$

Therefore, according to Cramer's rule, the solution is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 57/2 \\ -23/2 \end{bmatrix}$$

6. Properties of Determinants: T/F Mark each statement true or false, and give a reason to justify your answer. (**You will not get credit for any correct answer unless you give a justification.**) For all items, A and B are assumed to be $n \times n$ matrices.

_______ a. If the columns of A are independent then $\det A = 1$.

This is FALSE. Independent columns indicate a nonzero determinant, but not necessarily a determinant of 1. For example, a 2x2 diagonal matrix with 2 and 3 on the diagonal has independent columns and the determinant is 6, not 1.

_______ b. If the rows of A are dependent then $\det A = 0$.

This is TRUE – it is one of the results stated in the Determinants handout.

_______ c. $\det(AB) = \det(BA)$.

This is True. We know that det(*AB*) = det(*A*)det(*B*) and det(*BA*) = det(*B*)det(*A*). But since det(*A*) and det(*B*) are real numbers, we also know that det(*A*)det(*B*) = det(*B*)det(*A*). So we have

det(*AB*) = det(*A*)det(*B*) = det(*B*)det(*A*) = det(*BA*).

7. Computing determinants. Compute the determinant of each of the following, showing your work or stating a property of determinants that justifies your answer.

$$\begin{bmatrix} 1 & 8 & -2 & 3 & 5 \\ 0 & 2 & 7 & -1 & 3 \\ 0 & 0 & -1 & 1/2 & 1 \\ 0 & 0 & 0 & 4 & .8 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

This is a triangular matrix, so the determinant is the product of the diagonal entries: (1)(2)(-1)(4)(5)=-40.

$$\begin{bmatrix} -6 & 0 & 12 & 2 & 6 \\ 7 & 0 & 7 & 1 & 5 \\ 2 & 0 & 11 & -4 & 1 \\ 6 & 0 & -3 & 1 & 8 \\ 1 & 0 & 2 & 7 & 3 \end{bmatrix}$$

This matrix has a column of all 0's. That means the columns are not linearly independent, so the matrix is not invertible, so its determinant is 0. Or, if we compute the determinant by the method of minors, and choose all of our elements from the second column, then we see that all the coefficients will be 0 so the determinant is 0.

10. Special subspaces. For this problem, use the following matrices:

$$A = \begin{bmatrix} 2 & -3 & -7 & -1 & 7 \\ 1 & 0 & -2 & 1 & 2 \\ 0 & -2 & -2 & -2 & 2 \\ 1 & 1 & -1 & -1 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B was obtained by performing row operations on the matrix A. Using this information, find a basis for $\cdots$

a. the column space of A

A basis for the column space is given by the first, second, and fourth columns of *A*. We know that these are linearly independent because the corresponding columns of *B* are linearly independent, and because row operations do not change linear dependencies between the columns. Now we know that the column space of *A* is spanned by the 5 columns of *A*. But we can see in *B* that the third column is a linear combination of the first 2, and that the fifth column is a linear combination of columns 1, 2, and 4. The same must be true of the columns of *A*. So we see that the 3rd and 5th columns of *A* are dependent on columns 1 2 and 4, and can thus be eliminated without changing the set spanned by the columns. So we see that columns 1, 2, and 4 are linearly independent and span the column space. That makes them a basis for the column space. Alternative justification: We have seen a result that says the pivot cols of *A* form a basis for the column space.

b. the row space of A

The nonzero rows of *B* form a basis for the row space. We can see that they are linearly independent, and they clearly span the row space of *B*. Since that is the same as the row space of *A*, we can see that these rows are a basis for the row space of *A*. Alternate justification: we saw in class that the nonzero rows of the rref of *A* always constitute a basis for the row space of *A*.

c. the null space of A

The null space of *A* is the set of solutions to the homogeneous equation $Ax = 0$. Looking at *B*, we see that it is in rref form, so this is the rref of *A*. Solving for the basic variables, we find

$$x_1 = 2x_3 - x_5$$

$$x_2 = -x_3 + 2x_5$$

$$x_4 = 0x_3 - x_5.$$

This leads to the general solution of the homogeneous equation as

$$[x_1\ x_2\ x_3\ x_4\ x_5]^T = x_3\,[2\ \ {-1}\ \ 1\ \ 0\ \ 0]^T + x_5\,[{-1}\ \ 2\ \ 0\ \ {-1}\ \ 1]^T$$

and that shows that the two vectors on the right side of the equation are a spanning set for the null space. But they are also independent, as we can observe by looking at the entries in positions 3 and 5 – neither can be a multiple of the other. Since they are independent and span the null space, they are a basis for the null space. Alternate explanation: we saw in class that when we use the above method to find a vector parametric form for the solutions to the homogeneous equation, the vectors appearing in the parametric vector form always constitute a basis for the null space.

11. Basis and dimension results. Circle the correct response to complete each statement below, and **ALSO** state a theorem or give a reason that justifies your choice.

a. If V is a 7 dimensional vector space and S is a set of 10 vectors, then the elements of S _______ be linearly independent.

Circle one: **must** **might** **cannot.**

Justifying Theorem or Reason:

In a space of dimension *n* any set of more than *n* vectors is dependent. (That is a theorem from the book).

b. If V is a 4 dimensional vector space, and if the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{x}$ are linearly independent, then they _______ be a spanning set for V.

Circle one: **must** **might** **cannot.**

Justifying Theorem or Reason:

The four vectors are independent, so they form a basis for the subspace that they span. But that is then a four dimensional subspace of *V*, and therefore must actually equal *V*. Therefore, the given vectors are a basis for *V* and so must be a spanning set.

c. In a vector space V, if the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ span V, then the dimension of V must be ___ 3.

Circle one: $<$ $\leq$ $=$ $\geq$ $>$

Justifying Theorem or Reason:

Either the vectors are independent or they are not. If they are independent, then they are a basis for *V* and so dim *V* = 3. If they are dependent, we can eliminate one or more to leave a spanning set that is independent. That will still be a basis for *V*, but now with fewer than 3 elements. In this case the dimension of *V* is less than 3. This shows that across all possibilities, dim $V \le 3$.

12. Difference Equation Model. Consider a mouse population in a large wheat field. We consider the population to be subdivided into two categories, children and adults. The children are mice up to an age of two months, and the adults are mice older than two months.
We will model this situation with a sequence of vectors $\mathbf{v} = [a\ \ b]^T$, where *a* is the number of children and *b* is the number of adults. The terms of the sequence are denoted $\mathbf{v_0}, \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \ldots$ where $\mathbf{v_0}$ is the initial population vector, $\mathbf{v_1}$ is the population vector after 1 months, $\mathbf{v_2}$ is the population vector after 2 months, and so on.
We make the following assumptions:

i. Each month 30% of the children become adults, 30% of the children remain children, and 40% of the children die.
ii. Each month, the adults reproduce at a rate of 6 pups per breeding pair. That means a set of *x* adults will produce approximately 3x babies each month.
iii. Each month, 15% of the adults die.

Based on these assumptions each new vector $\mathbf{v}_{n+1}$ can be obtained by multiplying the preceding vector x by a fixed matrix *A*.

A. Find the matrix *A*, explaining your logic and/or showing your work.
Solution: Suppose there are *a* children and *b* adults at the start of a month. A month later, 30% of the children will have become adults, adding .3*a* to the adult population. And since 15% of the adults will die during the month, that will leave 85% or .85*b*. So, at the end of the month we will have .3*a* + .85*b* adults. By similar logic, 30% of the children remain in that category, giving us .3*a* children at the end of the month, and the reproduction of the *b* adults produces 3*b* new children. Thus at the end of the month there will be a total of .3*a* + 3*b* children. This shows that in one month, the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ becomes $\begin{bmatrix} .3a + 3b \\ .3a + .85b \end{bmatrix} = \begin{bmatrix} .3 & 3 \\ .3 & .85 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$. So the matrix *A* is $\begin{bmatrix} .3 & 3 \\ .3 & .85 \end{bmatrix}$.

B. In a similar model based on slightly different data, it was found that $A \begin{bmatrix} 12 \\ 5 \end{bmatrix} = 1.6 \begin{bmatrix} 12 \\ 5 \end{bmatrix}$. In that model, if there are initially 240 children and 100 adults, describe the sequence of population vectors produced by the model.
Solution: We see that $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ is an eigenvector of *A* with corresponding eigenvalue 1.6. The initial vector is $\mathbf{v_0} = \begin{bmatrix} 240 \\ 100 \end{bmatrix} = 20 \begin{bmatrix} 12 \\ 5 \end{bmatrix}$, so it and all its multiples are also eigenvectors. This tells us that every time we multiply by *A*, it has the same effect as multiplying by 1.6. Thus the sequence of population vectors will be $\mathbf{v_0}$, $1.6\mathbf{v_0}$, $1.6^2\mathbf{v_0}$, etc. After *n* months, the population vector will be $1.6^n\mathbf{v_0}$.

13. For all of the parts of this problem, $A = \begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}$.

A. Is $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ an eigenvector for this matrix? If so, find its eigenvalue, showing your work. If not explain how you know it is not.

Solution: We multiply $\begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 30 \\ 18 \end{bmatrix} = 6\begin{bmatrix} 5 \\ 3 \end{bmatrix}$. This shows that $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is an eigenvector with corresponding eigenvalue 6, because $\begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is a nonzero vector.

B. Is $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ an eigenvector for this matrix? If so, find its eigenvalue, showing your work. If not explain how you know it is not.

Solution: We multiply $\begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 39 \\ 15 \end{bmatrix}$. Is this a multiple of $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$? If so, we would have $\begin{bmatrix} 39 \\ 15 \end{bmatrix} = c\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 3c \\ 6c \end{bmatrix}$, and hence the system of equations

$$3c = 39$$
$$6c = 15.$$

Doubling the first equation we obtain the equivalent system

$$6c = 78$$
$$6c = 15,$$

which is evidently an inconsistent system. Thus there is no solution c. Since $\begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is not equal to a scalar multiple of $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$, we see that $\begin{bmatrix} 3 \\ 6 \end{bmatrix}$ is not an eigenvector of $\begin{bmatrix} 3 & 5 \\ 3 & 1 \end{bmatrix}$.

C. Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ an eigenvector for this matrix? If so, find its eigenvalue, showing your work. If not explain how you know it is not.

Solution: Since we know that 6 is an eigenvalue of the matrix, the corresponding eigenspace includes the zero vector. That is, the zero vector is one of the eigenvectors associated with the eigenvalue 6.

D. Is 4 an eigenvalue for this matrix? If so, find two of its eigenvectors. If not, explain how you know.

Solution: We want to know whether there is a nonzero vector solution $\mathbf{x}$ to the equation $A\mathbf{x} = 4\mathbf{x}$. This is equivalent to the homogeneous equation $(A - 4I)\mathbf{x} = \mathbf{0}$. So we consider the coefficient matrix $A - 4I = \begin{bmatrix} -1 & 5 \\ 3 & -3 \end{bmatrix}$. The determinant is $3 - 15 \neq 0$, so the matrix is invertible. This tells us that the rref if the identity matrix, and that there are no nontrivial solutions to the homogeneous equation. Thus, there are no nonzero solutions to the equation $A\mathbf{x} = 4\mathbf{x}$, and this shows that 4 is not an eigenvalue.

14. For the matrix $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$ find all the eigenvalues, and for one of the eigenvalues, find a complete description of the corresponding eigenspace.

Solution: To find the eigenvalues, we set up and solve the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = -\lambda(1-\lambda) - 2 = 0$$

$$-\lambda + \lambda^2 - 2 = 0$$

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2, -1.$$

This shows that the eigenvalues of the matrix A are 2 and -1. These values can also be found using the quadratic formula as an alternative to factoring.

Let's find the eigenvectors that correspond to an eigenvalue of 2. We need to find solutions to the equation $A\mathbf{x} = 2\mathbf{x}$, which can be re-expressed as $(A - 2I)\mathbf{x} = 0$. That gives us the homogeneous matrix equation $\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We can solve this by finding the rref of the coefficient matrix. As a preliminary step, swap the two rows and then add the first to the second. That produces $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$. This shows that our matrix equation is equivalent to

$\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and so to the single equation $2x - y = 0$, as well as to this single equation: $y = 2x$. Therefore, the solutions to our equation are all the vectors of the form $\begin{bmatrix} x \\ 2x \end{bmatrix}$. In parametric form, we say the solution set consists of all vectors $x\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where x is a free parameter. In other words, the eigenvectors for the eigenvalue 2 consist of all the scalar multiples of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. This is also the linear span of the set $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$, and is also nul A (the null space of A). It is this null space that we call the eigenspace corresponding to the eigenvalue 2.