Linear Algebra
Spring 2016
Exam 1 Name ______________________________

Instructions: For full credit, you must show work or give some explanation of how you reached each answer. You must communicate to me **how** you reached your answer. Your exam should have printing on 8 numbered pages. The last page is blank and may be used for scratch work.

1. Definitions (15 points). Give a mathematically correct definition for each term or expression below:

A. *Inconsistent* system of linear equations

An inconsistent of linear equations is one for which no solutions exist.

B. $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \cdots \mathbf{v}_n\}$ is a *linearly independent* set of vectors.

There are two equally valid answers:
(1) The set of vectors in linearly independent provided no vector in the set is equal to a linear combination of the other vectors in the set.
(2) The set of vectors is linearly independent provided the vector equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \quad + c_n\mathbf{v}_n = \mathbf{0}$$

has *only* the trivial solution $c_1 = c_2 = \quad = c_n = 0$.

C. A is an elementary matrix.

Matrix A is an elementary matrix provided it can be obtained by performing a single row operation on the identity matrix. An elementary row operation is one of the following: (1) multiply one row by a nonzero scalar, (2) interchange two rows, or (3) add a multiple of one row to another row, thus replacing that other row.

2. (8 points) Put the following system of equations into a matrix equation form. [Note: matrix equation form is not the same thing as an augmented matrix.]

$$\begin{aligned} x + 2z &= 1 \\ y + 2z &= 0 \\ x - 1 &= y \\ 2z - 3y + x &= -5 \end{aligned}$$

Rewrite the equations so that the variables are in consistent columns, including zero coefficients where necessary:

$$x + 0y + 2z = 1$$
$$0x + y + 2z = 0$$
$$x - y + 0z = 1$$
$$x - 3y + 2z = -5.$$

The matrix equation is then

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -5 \end{bmatrix}$$

3. (8 points) Put the equation below into the form of a vector equation.

$$\begin{bmatrix} 0 & -3 & 2 & 1 \\ 2 & 0 & 6 & 1 \\ 1 & -1 & 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 13 \end{bmatrix}$$

$$x \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix} + z \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix} + w \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 13 \end{bmatrix}$$

4. (9 points) One theorem from the book is partially stated below. Fill in the missing parts of the statement.

Theorem: Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A they are all true statements are they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b.

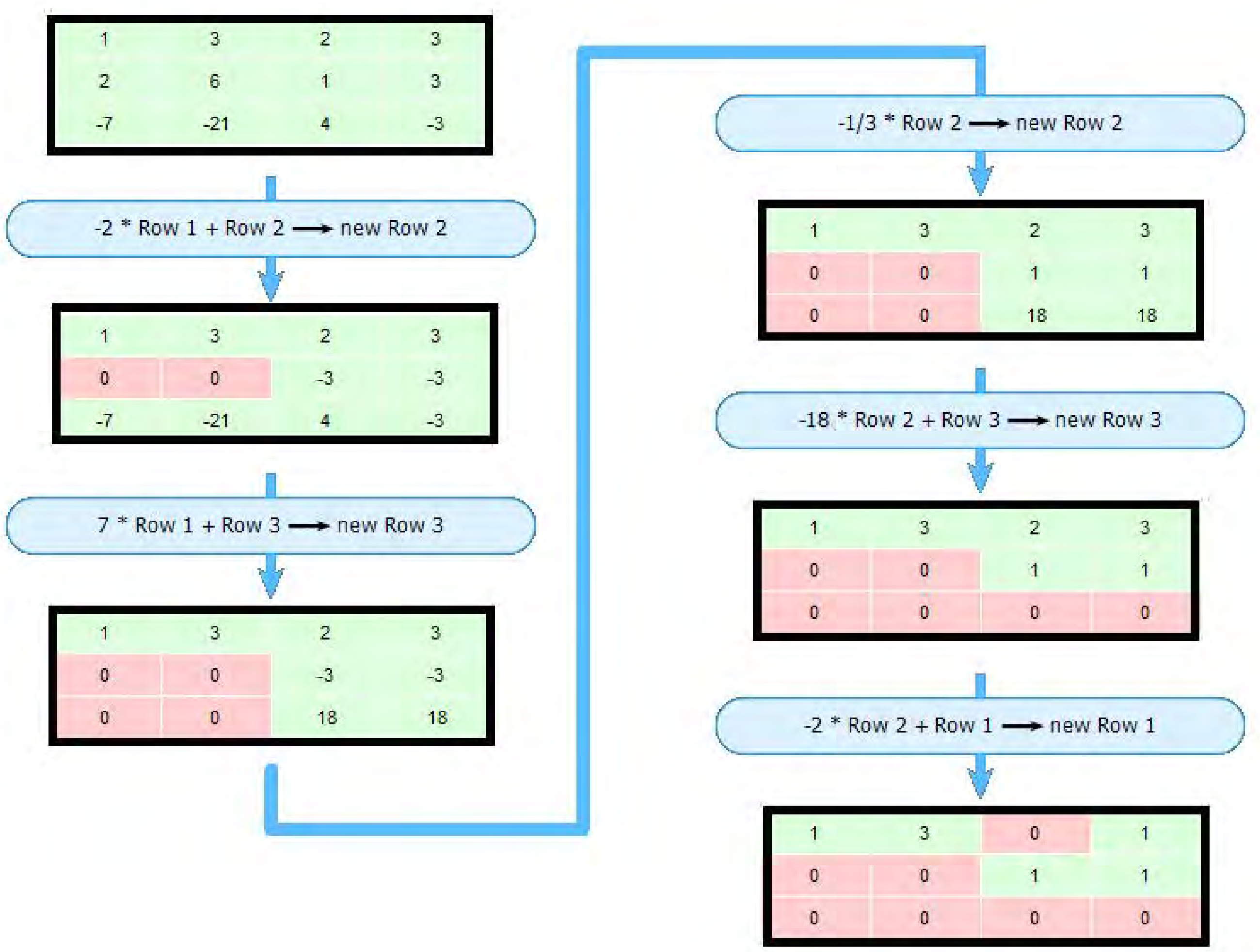
See Theorem 4 on page 43.

c.

d.

5. (16 points) Find the reduced row echelon form for the matrix below, showing all your steps. In particular, for each step indicate what row operation(s) have been performed.

$$\begin{bmatrix} 1 & 3 & 2 & 3 \\ 2 & 6 & 1 & 3 \\ -7 & -21 & 4 & -3 \end{bmatrix}$$



6. (12 points) In each part below, indicate whether or not the given matrix is in reduced row echelon form (RREF). For those which are **NOT** in RREF, tell **WHY**.

A. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is in rref.

B. $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

This is not in rref because there is a row of zeros not at the bottom.

C. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

This is not in rref because the 1 in row 2 is a leading entry, but there is a nonzero entry above it.

D. $\begin{bmatrix} 1 & 0 & 2 & 5 & 0 \\ 0 & 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

This is in rref.

7. (14 points) The augmented matrix for a system of equations is reduced by row operations to the following form:

$$\left[\begin{array}{ccccc|c} 1 & 5 & 0 & 0 & -8 & -11 \\ 0 & 0 & 1 & 0 & 23 & -2 \\ 0 & 0 & 0 & 1 & 7 & 13 \end{array}\right]$$

Give a parametric description of the solution set to the system of equations. Also give the parametric vector form for the solutions when the system is viewed as a vector equation.

The rref shown can be re-expressed as the following system of equations:

$$x_1 + 5x_2 - 8x_5 = -11$$
$$x_3 + 23x_5 = -2$$
$$x_4 + 7x_5 = 13$$

We solve for the first variable in each equation, finding that all solutions of the system satisfy

$$x_1 = -11 - 5x_2 + 8x_5$$
$$x_3 = -2 - 23x_5$$
$$x_4 = 13 - 7x_5$$

where x_2 and x_5 are free parameters. This a parametric description of the set of solutions.

For the vector version, we represent a solution vector in the form $[x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5]^T$. Substituting the expressions above for the variables x_1, x_3, and x_4 shows that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -11 - 5x_2 + 8x_5 \\ x_2 \\ -2 \quad - 23x_5 \\ 13 \quad - 7x_5 \\ x_5 \end{bmatrix} = \begin{bmatrix} -11 \\ 0 \\ -2 \\ 13 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 8 \\ 0 \\ -23 \\ -7 \\ 1 \end{bmatrix}$$

This a parametric vector form for the set of solutions to the vector equation for the given system of equations.

9. (10 points) Do the vectors below span $\mathbb{R}^3$? Justify your answer.

$$\begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 7 \\ 11 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 3 \\ 26 \end{bmatrix}$$

No.

Because the first entry of each vector is 0, any linear combination of these vectors will also have a first entry of 0. In particular, no linear combination of these three vectors can equal $[1 \quad 0 \quad 0]^T$. This in one specific vector in $\mathbb{R}^3$ that is not a linear combination of the three given vectors, showing that these vectors do not span $\mathbb{R}^3$.

5. Invertible Matrix Theorem [**15 points**]. Let A be an $n \times n$ matrix. Complete the following conditions that are equivalent to A being invertible.

a. The columns of A ...

... are linearly independent.

Or, equally correct,

... span $\mathbb{R}^n$

b. The equation $A\mathbf{x} = \mathbf{0}$ has ...

... only the trivial solution, or has the unique solution $\mathbf{x} = \mathbf{0}$.