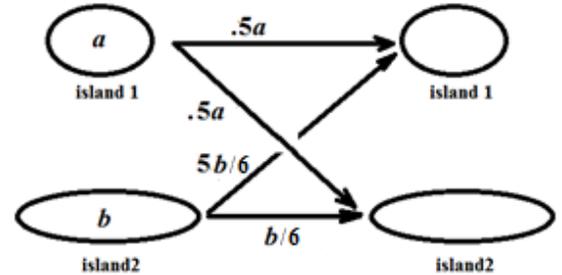


### Seagull Population Problem

Consider two islands populated by sea gulls. The total population remains constant but every month half of the gulls on island 1 move to island 2, while 5/6 of the gulls on island 2 move to island 1. The problem is to see what happens to the populations of the islands over time, given various possible initial populations.

We can calculate what happens conveniently using matrix algebra. Suppose there are initially  $a$  gulls on island 1 and  $b$  gulls on island 2. After 1 month half of the gulls on island 1, that is  $a/2$ , move to island 2, and  $a/2$  remain. Similarly, one sixth of the  $b$  island 2 seagulls stay on island 2, while 5/6 move to island 1. This situation is shown in a diagram at right. Combining these results we see that after one month there will be  $(1/2)a + (5/6)b$  gulls on island 1 and  $(1/2)a + (1/6)b$  gulls on island 2.



Now let us formulate this information using vectors. The initial population vector is  $[a \ b]^T$ , indicating  $a$  birds on island 1 and  $b$  birds on island 2. One month later the population vector is given by

$$\begin{bmatrix} 1/2 \cdot a + 5/6 \cdot b \\ 1/2 \cdot a + 1/6 \cdot b \end{bmatrix} = a \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + b \begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1/2 & 5/6 \\ 1/2 & 1/6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Thus, we can model the change in the population distribution after one month by the transformation

$$\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{bmatrix} 1/2 & 5/6 \\ 1/2 & 1/6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

In other words, each month, the distribution vector gets multiplied by the matrix  $A = \begin{bmatrix} 1/2 & 5/6 \\ 1/2 & 1/6 \end{bmatrix}$ .

To see what happens over time, you can start with some distribution vector, say  $[6000 \ 6000]^T$ , and repeatedly multiply by matrix  $A$ . You will have to do a number of experiments with different starting vectors and multiply each by  $A$  several times to see the patterns that emerge. For this purpose, Freemat is very convenient. Use the steps below to do your computations efficiently. At each step, type what follows --> and hit enter.

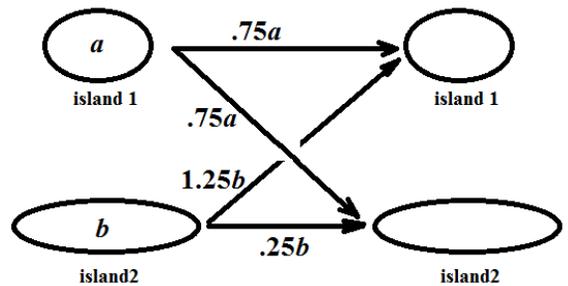
What to do:	Freemat Display
Define the matrix A	--> A = [1/2 5/6;1/2 1/6] A = 0.5000    0.8333 0.5000    0.1667
Define an initial vector x	--> x = [6000; 6000] x = 6000 6000
In one command, change x into Ax, and show the result	--> x = A*x x = 8000 4000

<p>Now you can repeat step 3 by pressing the up arrow key ↑ and <b>enter</b>. Do that as many times as you wish.</p>	<pre>--&gt; x = A*x x = 7333.3 4666.7</pre>
--	---

Experimentally, we find that in the long run the population vector approaches a constant  $[7500 \ 4500]^T$ . Further experimentation reveals that from any initial vector, this model produces a sequence of vectors converging to a limiting vector, and that the limit is always a multiple of  $[7500 \ 4500]^T$ . On the other hand, if we start with an initial population vector of  $[7500 \ 4500]^T$ , we obtain a constant sequence. That is, each new vector is identical to the preceding one. This is a form of equilibrium. The migration between the birds on the two islands is exactly in balance, so that the populations on each island remain constant. Mathematically, this is reflected in the fact that  $A[7500 \ 4500]^T = [7500 \ 4500]^T$ . More generally, if  $\mathbf{v}$  is any vector parallel to  $[7500 \ 4500]^T$ , then  $A\mathbf{v} = \mathbf{v}$ .

### A Revised Model

Let us modify the assumptions of the model. Suppose that the bird population is growing by 50% every year. (This is an unrealistic model in most ecosystems, but we will adopt it as a matter of convenience. The point is to get an idea of what might happen with an increasing population.) Repeating our earlier steps, suppose again there are initially  $a$  gulls on island 1 and  $b$  gulls on island 2. After 1 month, island 1's population will have grown to  $1.5a$ , but half that amount, or  $.75a$  will have moved to island 2. Similarly, the island 2 population will have grown to  $1.5b$ , of which  $5/6$  (or  $1.25b$ ) will have moved to island 1, leaving  $.25b$  on island 2. This can again be presented in a diagram, as shown at right. In this version of the model we see that after one month there will be  $.75a + 1.25b$  gulls on island 1 and  $.75a + .25b$  gulls on island 2.



Once again we formulate this information using vectors. The initial population vector is still  $[a \ b]^T$ , and one month later the population vector is given by

$$\begin{bmatrix} .75a + 1.25b \\ .75a + .25b \end{bmatrix} = a \begin{bmatrix} .75 \\ .75 \end{bmatrix} + b \begin{bmatrix} 1.25 \\ .25 \end{bmatrix} = \begin{bmatrix} .75 & 1.25 \\ .75 & .25 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

This leads to a modified version of the earlier model. This time the change in the population distribution after one month is modeled by the transformation

$$\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{bmatrix} .75 & 1.25 \\ .75 & .25 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

In other words, each month, the distribution vector gets multiplied by the matrix  $A = \begin{bmatrix} .75 & 1.25 \\ .75 & .25 \end{bmatrix}$ .

As before, we can use Freemat to follow the course of this model numerically. This time let us begin with an initial population vector of  $[6000 \ 3000]^T$ , and repeatedly multiply by the new matrix  $A$ . The Freemat commands are shown below.

What to do:	Freemat Display
Define the matrix A	--> A = [.75 1.25; .75 .25] A = 0.7500      1.2500 0.7500      0.2500
Define an initial vector x	--> x = [6000;3000] x = 6000 3000
In one command, change x into Ax, and show the result	--> x = A*x x = 8250 5250
Now you can repeat step 3 by pressing the up arrow key ↑ and <b>enter</b> . Do that as many times as you wish.	--> x = A*x x = 12750 7500

This time, the population vectors do not approach a constant limit. But we should not expect that, with a population growing by 50% at each step. Although the actual number of birds on the islands will grow without bound, we can consider the percentage of birds on each island.

If the population vector at any point is  $[p \ q]^T$ , then the total population will be  $p+q$  and the percentages on each island can be represented as  $100p/(p+q)$  and  $100q/(p+q)$ . In order to put this into our Freemat computations, we can express  $p+q$  as the vector product  $[1 \ 1][p \ q]^T$ . Thus we can formulate the vector of percentages as follows:

$$\begin{bmatrix} \text{Island 1 \%} \\ \text{Island 2 \%} \end{bmatrix} = 100 \begin{bmatrix} p \\ q \end{bmatrix} / \left( [1 \ 1] * \begin{bmatrix} p \\ q \end{bmatrix} \right).$$

In the Freemat workspace, we denoted  $[p \ q]^T$  by **x**. So, we can obtain each new percentage vector by replacing **x** by **Ax** and then printing out  $100*x/([1 \ 1]*x)$ . This can be done with a command on one line of Freemat, as shown in the table below.

What to do:	Freemat Display
Define the matrix A	--> A = [.75 1.25; .75 .25] A = 0.7500      1.2500 0.7500      0.2500
Define an initial vector x	--> x = [6000; 3000] x = 6000 3000
In one command, change x into Ax, and then show the result of $100*x/([1 \ 1]*x)$	--> x = A*x x = 61.1111 38.8889
Now you can repeat step 3 by pressing the up arrow key ↑ and <b>enter</b> . Do that as many times as you wish.	--> x = A*x; 100*x/([1 1]*x) x = 62.9630 37.0370

Experimental results from this set up shows that from any initial population vector, the percentage vector approaches  $[62.5 \ 37.5]^T$ . We can relate this to the actual populations by realizing that if the total population is originally  $P$ , then after  $n$  months it will have grown to  $1.5^n P$ . Thus, if 62.5% of the population is on island 1, and 37.5% on island 2, the population vector would have to be  $1.5^n [0.625 \ 0.375]^T = 1.5^n [5/8 \ 3/8]^T$ .

And indeed, if we set  $\mathbf{x} = 1.5^n [5/8 \ 3/8]^T$ , then  $A\mathbf{x} = 1.5\mathbf{x} = 1.5^{n+1} [5/8 \ 3/8]^T$ . This shows that the percentage vector is held constant when we multiply the population vector by  $A$ . Mathematically, the key fact is that  $A[5/8 \ 3/8]^T = 1.5[5/8 \ 3/8]^T$ . This is reminiscent of what happened in the first model, if we interpret the equation  $A[7500 \ 4500]^T = [7500 \ 4500]^T$  as showing a case of multiplying by 1:  $A[7500 \ 4500]^T = 1 \cdot [7500 \ 4500]^T$ . Thus, for each model, the long term behavior converged to a vector with a special property: multiplying by  $A$  had the effect of multiplying by a particular constant – 1 in the first model and 1.5 in the second.

The two versions of this model are instances of a large category of models that arise in applications. The common mathematical idea involves a sequence of vectors that is propagated through repeated multiplication by a fixed matrix  $A$ . The vectors in the examples represent populations of birds on two islands, but something similar can happen any time there are several related variables that evolve over time. In this setting, the vector of variable values at a particular time represents the *state* of the system at that time, and the sequence of vectors represents successive states of the system. In the most general setting, each of the variables in the state vector can be found by operating on *all* the variables at an earlier state, and possibly on the amount of time between states. As a special case, we can develop a model where each variable in the state vector is a linear combination of the variables in the preceding state vector. That is equivalent to saying that there is a constant square matrix  $A$  (containing all the coefficients of the linear combinations determining all the variables), and multiplying that matrix by one state vector produces the next. In symbols, we write

$$A\mathbf{x}_n = \mathbf{x}_{n+1}, \tag{1}$$

where  $\mathbf{x}_n$  is the current state and  $\mathbf{x}_{n+1}$  is the succeeding state. Equation (1) is referred to as a *vector difference equation*, or more briefly, as a *difference equation* when it is clear from context that the  $\mathbf{x}$ 's are vectors.

In working with a model based on a vector difference equation, the following are typical questions of interest:

1. Can we find an equation expressing  $\mathbf{x}_n$  as a function  $n$ ?
2. How does the sequence of vectors depend on the choice of the initial vector?
3. Can we predict the future evolution of the system from properties of the matrix  $A$ ?

Notice that for the models presented above, concerning seagull populations, we did not find an equation for  $\mathbf{x}_n$  as a function  $n$ , but we did find that in the long run, both models predicted that the distribution of seagulls between the two islands would approach an equilibrium. Moreover, we found that the equilibrium did not depend on the initial state vector – the populations ended up with the same distribution between islands no matter what starting vector we considered. These are typical conclusions in difference equation models.

There is another aspect of our examples that is characteristic of difference equation models. In both versions, we found that a special vector emerged, closely connected to the equilibrium state of the system, and exhibiting a special relation to the matrix  $A$ . To be specific, for each of these  $\mathbf{x}$  vectors, we found  $A\mathbf{x}$  to be a scalar multiple of  $\mathbf{x}$ . Geometrically, multiplying these special vectors by  $A$  did not change the *direction* of the vectors, only the length. Vectors with this property are called *eigenvectors* of  $A$ . We will develop methods for finding eigenvectors, and using them to analyze difference equation models.

This handout concludes with two additional examples of difference equation models.

### Stratified Age Model

In the first set of examples we considered a population of birds that were divided into subpopulations according to the islands on which they lived. Now we will consider a different population of birds, but this time they will be divided into subpopulations on the basis of their ages. As described at the start of chapter 5 in our text, the birds in this case are owls, and they are classified as *juveniles*, *subadults*, and *adults*. Suppose that a population of owls in one region has 10,000 members, and that 2500 are juveniles (one year old or less), 500 are subadults (between 1 and 2 years old). That means 7000 must be adults. We put the subpopulations into a vector

$$\mathbf{x} = \begin{bmatrix} x_{\text{juv}} \\ x_{\text{sub}} \\ x_{\text{adult}} \end{bmatrix} = \begin{bmatrix} 2500 \\ 500 \\ 7000 \end{bmatrix}.$$

This vector tells us both the total number of owls, and also how they are distributed by age. This is the state vector for our difference equation model.

This model is based on a published research in the journal Conservation Biology (see our text for the full bibliographic citation). In the original paper, the model considered one state vector per year, with a predicted annual update computation. Each year, 18% of the juveniles would survive to become the next year's subadults. Therefore, the new value of  $x_{\text{sub}}$  would be given by

$$\text{new } x_{\text{sub}} = .18(\text{current } x_{\text{juv}}).$$

Similarly, 71% of the sub-adults would survive to join the adult population. Meanwhile, 94% of the adults would survive to remain in the adult population. This shows that the new value of  $x_{\text{adult}}$  would be given by

$$\text{new } x_{\text{adult}} = .71(\text{current } x_{\text{sub}}) + .94(\text{current } x_{\text{adult}}).$$

The next year's juveniles would all result from reproduction of the current year's adults. Assuming that the reproduction rate is 33%, that leads to the equation

$$\text{new } x_{\text{juv}} = .33(\text{current } x_{\text{adult}}).$$

These three equations can be combined into one matrix equation:

$$\text{new } \begin{bmatrix} x_{\text{juv}} \\ x_{\text{sub}} \\ x_{\text{adult}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & .33 \\ .18 & 0 & 0 \\ 0 & .71 & .94 \end{bmatrix} \left( \text{current } \begin{bmatrix} x_{\text{juv}} \\ x_{\text{sub}} \\ x_{\text{adult}} \end{bmatrix} \right).$$

This is again a vector difference equation. It tells us the next state vector is obtained through multiplying a constant matrix times the current state vector.

As before we can experiment numerically using Freemat. It is more difficult to discern a pattern for this example than it was for the seagull models. One idea is to look at how the total population varies year by year

by year. We can calculate the total population as  $x_{\text{juv}} + x_{\text{sub}} + x_{\text{adult}} = [1 \ 1 \ 1] \begin{bmatrix} x_{\text{juv}} \\ x_{\text{sub}} \\ x_{\text{adult}} \end{bmatrix} = [1 \ 1 \ 1] * \mathbf{x}$ .

The set up for computing these values in Freemat is shown below.

What to do:	Freemat Display
Define the matrix M	--> M = [0 0 .33; .18 0 0; 0 .71 .94] M = 0                    0                    0.3300 0.1800               0                    0 0                    0.7100              0.9400
Define an initial vector x	--> x = [2500; 500; 7000] x = 2500 500 7000
In one command, change x into Mx, and then show the result of [1 1 1]*x	--> x = M*x; [1 1 1]*x ans = 9695
Now you can repeat step 3 by pressing the up arrow key ↑ and <b>enter</b> . Do that as many times as you wish.	--> x = M*x; [1 1 1]*x ans = 9542.8

Repeated application of this difference equation shows that the population size decreases every year. Extrapolating from these results we can conjecture that in the model the population will die out. Using Freemat we can verify this in the following way. Starting with an initial population vector  $\mathbf{x}$ , we know that the next year's vector will be  $M\mathbf{x}$ . For the following year we have to multiply by  $M$  again, producing  $M^2\mathbf{x}$ . For the next year the vector will be  $M^3\mathbf{x}$ . As this pattern suggests, after  $n$  years the population will be  $M^n\mathbf{x}$ , where  $\mathbf{x}$  is the initial population vector in all of these equations. Thus, if we want to project 100 years into the future, we can calculate  $M^{100}\mathbf{x}$ . Freemat will compute this if you enter  $M^{100}\mathbf{x}$ . The result is 1887.1. After 200 years the model predicts a population of 360.9. And after 300 years the prediction is 69.

The possible extinction of a population of owls after more than 200 years probably does not seem to be much cause for concern. But the point of this example is merely to illustrate how the model works. The initial population vector was made up for convenience (though the matrix  $M$  is based on the original paper). If there had initially been only 1000 owls instead of 10,000, then the population would die off sooner.

As explained above, the entry of .18 in the second row of  $M$  reflects a very low survival rate for the juvenile owls. Would the model's predictions change if that survival rate were higher? Changing the value from .18 to .50 results in a model where the population size grows slowly, instead of decreasing, and approaches a stable distribution of 22% juveniles, 10% subadults, and 68% adults. We will see that these conclusions can be derived by analyzing the eigenvectors of the different versions of the  $M$  matrix.

### A Vector Difference Equation for the Fibonacci Numbers

The Fibonacci numbers are the terms of the sequence 0, 1, 1, 2, 3, 5, 8, 13, ... in which each the starting terms are 0 and 1, and each term thereafter is the sum of the two preceding terms. It is customary to refer to the terms as  $F_0, F_1, F_2$ , etc. Using this notation,  $F_6 = 8$  and  $F_9 = 55$ . For some reason, these numbers have attracted the attention of math professionals and hobbyists, teachers and students, and even intruded into popular culture. For example, they play a central role in the plot of the novel *The da Vinci Code*. They are also connected with the idea of vector difference equations.

To reveal the connection, we create a sequence of vectors by pairing up the Fibonacci numbers, as follows:

$$\mathbf{V}_0 = \begin{bmatrix} F_0 \\ F_1 \end{bmatrix}, \quad \mathbf{V}_1 = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}, \quad \mathbf{V}_2 = \begin{bmatrix} F_2 \\ F_3 \end{bmatrix}, \quad \dots, \quad \mathbf{V}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}.$$

We can calculate these numerically, for example  $\mathbf{V}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ , but for what we will consider, it is more useful to leave the entries of the vectors as  $F_1, F_2$ , etc.

Because we are working toward a difference equation, it is worthwhile to ask how each  $\mathbf{V}$  vector leads to the next. For example, notice that

$$\mathbf{V}_5 = \begin{bmatrix} F_5 \\ F_6 \end{bmatrix} = \begin{bmatrix} F_5 \\ F_4 + F_5 \end{bmatrix} = \begin{bmatrix} 0F_4 + 1F_5 \\ 1F_4 + 1F_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_4 \\ F_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{V}_4.$$

We can write a similar line for any of the  $\mathbf{V}$  vectors, showing that the following equation holds for any  $n$ :

$$\mathbf{V}_{n+1} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{V}_n.$$

Using the same logic as in the previous example, we observe that

$$\mathbf{V}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{V}_0, \quad \mathbf{V}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{V}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^2 \mathbf{V}_0, \quad \mathbf{V}_3 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{V}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^3 \mathbf{V}_0, \quad \text{etc.}$$

In fact, for any positive integer  $n$  we can write

$$\mathbf{V}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \mathbf{V}_0 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Now multiply both sides of this equation by  $\begin{bmatrix} 1 & 0 \end{bmatrix}$ . On the left we have  $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = F_n$ . On the right we have  $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Thus we have derived the following formula for the  $n$ th Fibonacci number:

$$F_n = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (2)$$

That is not particularly useful for actual computation without some efficient method for finding powers of the matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ . But it turns out that just such a method emerges out of an understanding of eigenvectors. We will see how that all develops in the next part of the course. Then we will be able to use equation (2) to derive what is called the Binet formula for the Fibonacci numbers. That says

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^n - \left( \frac{1 - \sqrt{5}}{2} \right)^n \right].$$

This is an amazing formula! In the first place, it is incredible that a simple sequence of whole numbers could be described by such a complicated expression. It isn't at all obvious that using the expression on the right must always produce whole numbers. In the second place, even if the formula is correct, how on Earth could anyone ever figure it out? And finally, what has any of this got to do with the matrix  $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ ? These mysteries will all be explained when we learn about the topic of eigenvectors.

### Exercises

1. Use Freemat to experiment with results of the first seagull model. Start with each initial population vector shown in parts a through c below. In each case, generate successive population vectors using the procedure  $\begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{bmatrix} 1/2 & 5/6 \\ 1/2 & 1/6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$ . Decide whether the results seem to be approaching a limit. If they do, is the limit vector parallel to  $\begin{bmatrix} 7500 & 4500 \end{bmatrix}^T$ , as stated in the reading?
  - a. Initial population vector is  $\begin{bmatrix} 3000 & 9000 \end{bmatrix}^T$ .
  - b. Initial population vector is  $\begin{bmatrix} 8000 & 4000 \end{bmatrix}^T$ .
  - c. Initial population vector is  $\begin{bmatrix} 11000 & 1000 \end{bmatrix}^T$ .

2. Make up your own version of the first seagull model. Assume that the total population neither grows nor shrinks, and that the gulls move from one island to another. But instead of using the assumptions in the reading, that every month half the birds on island 1 move to island 2 and  $5/6$  of the birds on island 2 move to island 1, make up your own fractions. Then repeat the investigation described in the reading, and see whether you find similar results.
3. Repeat the analysis of the second seagull model using the initial population vectors as in problem 1. Recall that we used Freemat to compute successive vectors  $\mathbf{x}$  in the model, and then print out the vector of *percentages*, computed as  $100*\mathbf{x}/([1\ 1]*\mathbf{x})$ .
4. The description of the owl population model stated that the owls die out in the original version of the model, but will grow slowly if the matrix entry of .18 is changed to .50. That entry is the survival rate for juveniles: it represents the percentage of juveniles who survive to become subadults. Use Freemat to confirm that when this survival rate is .50, the owl population will grow larger over time. Then repeat your investigation taking the survival rate to be first .30 and then .40. Does the owl population die out or survive in these variations?
5. Continuing the preceding problem, it is reasonable to expect that there is a critical survival rate for juvenile owls that marks the transition between a population that shrinks and one that grows. That is, if the survival rate is too small, the owls will die out. If it is large enough, the population will grow without any maximum value. So somewhere in the middle there should be a survival rate for which the population remains constant, or at least approaches a constant value. Using Freemat to experiment with a variety of survival rates, see if you can discover what the critical value is.
6. We saw in the second seagull model that over time, the populations on the islands approached a constant distribution, in the sense that the percentage of seagulls on each island became constant. Another way of saying the same thing is this: over time the slope of the population vector becomes constant. Here, we define the slope of the vector  $[a\ b]^T$  to be  $b/a$ . Use Freemat to explore whether the Fibonacci example has this same property. That is, starting with  $\mathbf{x} = [0\ 1]^T$ , generate the vectors of the sequence by repeatedly computing  $\mathbf{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \mathbf{x}$ , and then printing out  $[0\ 1]*\mathbf{x}/([1\ 0]*\mathbf{x})$ . Do you find that the slopes of the successive  $\mathbf{x}$ 's approach a limit? If so, what is it? [Comment: the slope of the  $n$ th vector is  $F_{n+1}/F_n$ . So if you find that the slopes approach a limit, that says that the ratio of successive Fibonacci numbers becomes essentially constant as  $n$  goes to infinity.]