Analyzing Linear Dependencies

FACT:

Let A and B be row equivalent matrices. Remember that means that you can get one from the other by using row operations. Then any linear dependence among the columns of A also holds for the columns of B. For instance, if the third column of A is equal to twice the first column minus the second column, then the same is true for B. To be more precise: Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \cdots \mathbf{a}_n$, and the columns of B by $\mathbf{b}_1, \mathbf{b}_2, \cdots \mathbf{b}_n$. Then, for scalars $c_1, c_2, \cdots, c_n$, if $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n$ equals $\mathbf{0}$, we can also be certain that $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + \cdots + c_n\mathbf{b}_n = \mathbf{0}$

APPLICATION:

The fact above can be used to analyze dependencies among a given set of vectors. Create a matrix using the given vectors as columns. Then use row operations to reduce that matrix to reduced echelon form. Linear dependencies among the columns of the reduced form will be evident by inspection. These same dependencies hold for the original matrix.

Example: Determine if the following vectors are dependent, and if they are, describe the dependencies:

$$\begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 3 \\ 3 \\ -2 \\ -2 \\ 6 \end{bmatrix} \quad \begin{bmatrix} -3 \\ 3 \\ 10 \\ 19 \\ 3 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 5 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 6 \\ -6 \\ 3 \\ -4 \end{bmatrix}$$

Solution:

I entered the given columns in FREEMAT and used the rref command. Here is what came out:

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A =

     1     3    -3     1     1
     3     3     3     0     6
     2    -2    10     2    -6
     5    -2    19     0     3
     5     6     3     5    -4

rref(A)

ans =

     1     0     3     0     1
     0     1    -2     0     1
     0     0     0     1    -3
     0     0     0     0     0
     0     0     0     0     0
```

In the reduced echelon form for the matrix, it is obvious that the third column is equal to 3 times column 1 minus 2 times column 2. That same fact must hold for the original set of vectors. Check that the third vector is 3 times the first vector minus 2 times the second. It is also clear in the reduced echelon form that the column 5 equals column 1 + column 2 - 3(column 4). That same statement must hold for the original matrix. Finally, the three pivot columns of the reduced echelon form are independent, and that should be clear to you. So, to summarize the results: in the original set of 5 vectors, the third vector is a linear combination of the first 2; the fifth is a linear combination of the first, second, and fourth; and the first, second, and fourth vectors form an independent set.

EXPLANATION:

The reasoning behind the fact above is a good example of how using all of our different view points for linear equations leads to interesting conclusions. Here is the argument.

First, note that a linear dependence among a set of vectors can be expressed as a vector equation:

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_n\mathbf{a}_n = \mathbf{0}$$

Second, rewrite that as a matrix equation:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \mathbf{0}$$

This shows that a dependence among the vectors corresponds to a solution to a homogeneous matrix equation of the form $A\mathbf{c} = \mathbf{0}$.

Third, we know how to find solutions to matrix equations by using row operations. For the special case of a homogeneous equation, it is not necessary to include the augmenting column because that column of all zeros will not be changed by any row operations. The key point here is this: the row operations do not change the solutions to the system. We know this from the simultaneous equations point of view. Think of that matrix equation as a set of simultaneous equations in the n unknowns c_1 through c_n. Each row operation amounts to an algebraic rearrangement of these equations that doesn't change the solutions. This shows that after we do a set of row operations and change the original matrix A to a new matrix B, the solutions to the homogeneous equation for the two matrices will be identical.

Conclusion: combining all of the three points above, we see that any linear dependence among the columns of the original matrix A must also hold among the columns of B, and vice versa.

Discussion: An important point is illustrated by the argument above. The different views we use, simultaneous linear equations, vector equations, and matrix equations, each has a conceptual setting that makes certain ideas obvious. The fact that row operations do not change solutions is obvious in the conceptual setting of simultaneous equations. It might not be so obvious in the setting of a matrix equation. The fact that a dependence among vectors is just a solution to a homogeneous equation is obvious in the conceptual setting of vector equations. It is less obvious in the setting of simultaneous equations. By combining the different views, we get to combine several of these obvious insights, and in the end we have a result that is far from obvious. This is a common kind of occurrence in linear algebra.