Linear Algebra Class Notesheet: 3/24/2017

Principle: λ is an eigenvalue of A iff $\det(A - \lambda I) = 0$

Example 1: find all eigenvalues of $A = \begin{bmatrix} -12 & -60 & 45 \\ 4 & 28 & -21 \\ 2 & 20 & -15 \end{bmatrix}$ by setting

$$\begin{vmatrix} -12-\lambda & -60 & 45 \\ 4 & 28-\lambda & -21 \\ 2 & 20 & -15-\lambda \end{vmatrix} = 0.$$

Using freemat:

```
--> poly(A)
```
computes coefficients of $p(\lambda) = | A - \lambda |$

```
--> roots(poly(A))
```
computes roots of $p(\lambda)$

We can use these with rref to find eigenvalues as follows:

```
--> A = [-12 -60 45;4 28 -21;2 20 -15]
A =  -12 -60  45
       4  28 -21
       2  20 -15

--> poly(A)
ans =  1.0000   -1.0000   -6.0000    0.0000

--> roots(poly(A))
ans = 3.0000
     -2.0000
      0.0000
```

These are the eigenvalues of A

```
--> Ident = eye(3)
Ident =  1 0 0
         0 1 0
         0 0 1
```

This defines the 3x3 identity matrix

```
--> A - 3*Ident
ans = -15 -60  45
        4  25 -21
        2  20 -18
```

This is $A - \lambda I$ for $\lambda = 3$. The null space of this matrix consists of all the eigenvectors for $\lambda = 3$. We can find the null space by reducing to rref.

```
--> rref(ans)
ans =  1  0  1
       0  1 -1
       0  0  0
```

Using our usual methods, we find the basis for the null space: $\{ [-1 \quad 1 \quad 1]^T \}$

```
--> A + 2*Ident
ans = -10 -60  45
        4  30 -21
        2  20 -13
```

This is $A - \lambda I$ for $\lambda = -2$. The null space of this matrix consists of all the eigenvectors for $\lambda = -2$. We can again find the null space by reducing to rref.

```
--> rref(ans)
ans =  1.0000    0         -1.5000
       0         1.0000    -0.5000
       0         0          0
```

Using our usual methods, we find the basis for this null space: $\{ [1.5 \quad .5 \quad 1]^T \}$

This same approach will not work when the eigenvalues are irrational numbers. The numerical computation of the eigenvalues only gives us approximate values, and the corresponding $(A - \lambda I)$ matrices will not have determinant exactly = 0 because we don't have the exact correct values of λ.

Example 2: find all eigenvalues of $A = \begin{bmatrix} 4 & 1 & 5 \\ -1 & 5 & 7 \\ 5 & -2 & 2 \end{bmatrix}$.

```
--> B = [4 1 5;-1 5 7;5 -2 2]
B = 4  1  5
   -1  5  7
    5 -2  2

--> poly(B)
ans =  1.0000  -11.0000   28.0000   -18.0000

--> roots(poly(B))
ans = 7.6458
      2.3542
      1.0000
--> lambda1 = ans(1)
lambda1 =  7.6458

--> B1 = (B-lambda1*Ident)
B1 = -3.6458    1.0000    5.0000
     -1.0000   -2.6458    7.0000
      5.0000   -2.0000   -5.6458

--> rref(B1)
ans =  1 0 0
       0 1 0
       0 0 1

--> B1(1:2,:)
ans =  -3.6458    1.0000    5.0000
       -1.0000   -2.6458    7.0000
```

Here $B - 7.6458I$ does NOT have determinant = 0 because 7.6458 is not exactly an eigenvalue. So when we rref $B1$ we get I indicating only trivial solutions to the homogeneous equation. That does not provide even approximate eigenvectors. Procedures for finding the eigenvectors approximately have been worked out, and freemat has a command for that: `[P,D]=eig(B)` will show you a matrix P with approximate eigenvectors in columns, and a matrix D with approximate eigenvalues for the columns

```
--> [P,D]=eig(B)
P = 0.6585   -0.4082    0.3376
    0.6680   -0.8165    0.8954
    0.3466    0.4082   -0.2902
D = 7.6458         0         0
         0    1.0000         0
         0         0    2.3542
```

Here, vector **v** is defined to be the first column of P.

```
--> v = [P(1,1);P(2,1);P(3,1)]
v = 0.6585
    0.6680
    0.3466
```

Here we compute $B\mathbf{v}$ and $\lambda_1\mathbf{v}$ and display them side by side. They agree to four decimal places, showing that **v** is approximately an eigenvector.

```
--> [B*v  lambda1*v]
ans =
    5.0349    5.0349
    5.1075    5.1075
    2.6497    2.6497
```