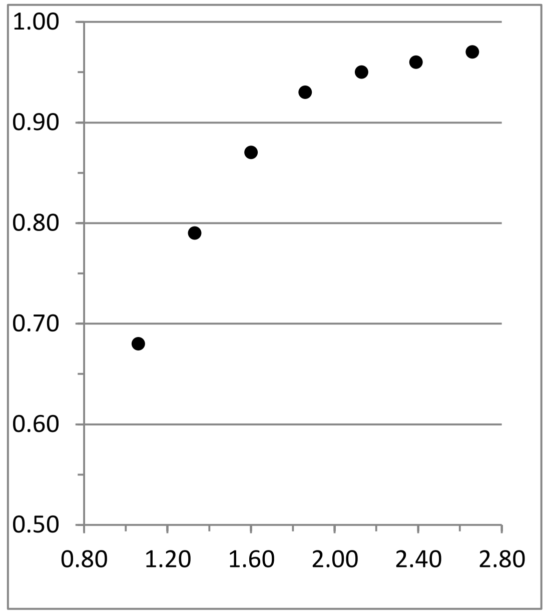
Work Sheet and Review of Least Squares and Eigen-Stuff

1. This problem uses real data from a math class experiment concerning the way an inline skater coasts to a stop on a level surface. Here is how the data were collected. A skater moves toward the starting line. As she crosses the starting line she begins to coast, and a stop watch is started. Then at various times data points are collected, each showing the skater's distance from the starting line and the corresponding time on the stop watch. Distances are recorded in meters and times in seconds. The data and a graph are shown below.

Time	1.06	1.33	1.6	1.86	2.13	2.39	2.66
Distance	0.68	0.79	0.87	0.93	0.95	0.96	0.97



The shape of the graph suggests that the data might be reasonably approximated by a function of the form

$$f(t) = A + \frac{B}{t}$$

with *A* and *B* suitable constants. Your task is to use a least squares approach to find the constants *A* and *B* that give the best approximation to the data in the table. Explain each step and show your work.

After you find your values of *A* and *B*, compute the values of $f(t)$ for each *t* in the table above. [Hint: you can do this as a matrix product of your design matrix times the least squares solution vector $\hat{\mathbf{x}}$.)

Next, create a plot showing the original data (distance vs time) as well as the *f* values for your optimal *A* and *B*. You can use excel for this, or use a plot command in freemat. If you wish to do the latter, create a column vector **v** that has the original time values, and a matrix **Y** with two columns: original distance data values in column 1, and the computed distance values in column 2. Then enter the command plot(**v**, **Y**).

Finally, compute the error vector (comparing the distance values in the data with the distance values predicted by the function *f*). What is the length of the error vector? What is the squareroot of the average squared error?

2. A biologist is studying a population of foxes. She subdivides the population into four categories, by age: ages 0 to 1 are in group 1, ages 1 to 2 in group 2, ages 2 to 5 in group 3, and all foxes older than 5 in group 4. At any given time, the distribution of foxes into these categories is represented by a vector, $\mathbf{p} = [\,g_1 \quad g_2 \quad g_3 \quad g_4\,]^T$. For example, if at one particular time there are 50 foxes in group 1, 40 in group 2, 20 in group 3, and 80 in group 4, the population vector at that time would be $\mathbf{p} = [50 \quad 40 \quad 20 \quad 80]^T$.

The biologist studies how many foxes in each group survive, and also how they reproduce. She finds that each year, approximately 60% of the group 1 foxes survive to enter group 2, and 70% of the group 2 foxes survive to enter group 3. For group 3, 24% move up to group 4, 48% remain in group 3, and 28% do not survive. For group 4, 80% survive and remain in group 4. Turning to reproduction, there are no offspring produced by groups 1 and 2, but each year the newborns from group 3 amount to approximately 1.6 times the number of group 3 foxes. The rate of reproduction is a bit higher for group four: their newborns amount to approximately 1.8 times the number of group 4 foxes.

Your task is to incorporate this information into a difference equation model with an equation of the form

$$\mathbf{p}_{k+1} = A\mathbf{p}_k ,$$

where A is a constant 4×4 matrix. The vector $\mathbf{p}_0$ is an initial population vector, with $\mathbf{p}_1$ the population vector after one year, $\mathbf{p}_2$, at the end of the second year and so on. The equation above says that population vector any year can be obtained by multiplying the preceding year's population vector by the matrix A.

a. Find the matrix A
b. Suppose that $\mathbf{p}_0 = [100 \quad 70 \quad 50 \quad 200]^T$. Determine the next 3 population vectors.
c. The matrix A has two real eigenvalues, approximately equal to 1.221 and 0.525, with corresponding eigenvectors approximately equal to $\mathbf{v}_1 = [809 \quad 398 \quad 376 \quad 215]^T$ and $\mathbf{v}_2 = [-42 \quad -48 \quad -752 \quad 656]^T$. If these were exact eigenvalues and eigenvectors, and if $\mathbf{p}_0 = .5\mathbf{v}_1 + .8\mathbf{v}_2$, what would $\mathbf{p}_{20}$ be? Using freemat, compare your prediction with what the actual value of $\mathbf{p}_{20}$ is.
d. Find a formula for $\mathbf{p}_k$, assuming the eigenvalues and eigenvectors given above are exact, and that $\mathbf{p}_0 = .5\mathbf{v}_1 + .8\mathbf{v}_2$.

3. A 5×5 matrix A has eigenvalues -1.5, 2, 2, 2, 1.7. To find eigenvectors for the eigenvalue 2, the

rref of $A-2I$ is calculated to be $\begin{bmatrix} 1 & 0 & -2 & 0 & 3 \\ 0 & 1 & -5 & 0 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

a. Give a description of all the eigenvectors corresponding to the eigenvalue 2.

b. Is the matrix A diagonalizable? Circle the correct response below, and then explain your reasons.

***A* is definitely diagonalizable**	***A* is definitely not diagonalizable**	**Cannot be determined from the given information**

4. Let A be an $m \times n$ matrix with $n < m$ and suppose that $A^{\mathrm{T}}A = I$. Then

a. Show that for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $\mathbf{u} \bullet \mathbf{v} = (A\mathbf{u}) \bullet (A\mathbf{v})$

b. Show that for any vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, $\mathbf{u} \perp \mathbf{v}$ if and only if $(A\mathbf{u}) \perp (A\mathbf{v})$.

c. Show that for any vector $\mathbf{v}$ in $\mathbb{R}^n$, $\| \mathbf{v} \| = \| A\mathbf{v} \|$.

d. Show that for any $\mathbf{v}$ in $\mathbb{R}^n$, the projection of $\mathbf{v}$ on the column space of A is given by $\mathbf{p} = AA^T\mathbf{v}$.

5. True False Items. Mark each item below as True or False. You should have a definite reason for each answer. Some of the false items are misstatements of facts or theorems we have covered. For these, you should be able to provide a corrected statement that is true.

a. Every invertible square matrix is diagonalizable.

b. If A is an $n \times n$ matrix and if **u**, **v**, and **w** are eigenvectors, then {**u**, **v**, **w**} is a linearly independent set.

c. If A is an $n \times n$ matrix and if $A^T = A$, then A is diagonalizable.

d. An $n \times n$ matrix A is diagonalizable if and only if there exists a basis for $\mathbb{R}^n$ in which every vector is an eigenvector of A.

e. If A is an $n \times n$ matrix and if A has n different real eigenvalues, then A is diagonalizable.

f. If A is similar to the diagonal matrix D then, for any k, $A^k = D^k$.

g. If A is an $n \times n$ matrix and if A has fewer than n different real eigenvalues, then A is NOT diagonalizable.

h. Every $n \times n$ triangular matrix is diagonalizable.

i. Suppose A is an $n \times n$ matrix and λ is and eigenvalue of A. If the dimension of the eigenspace for λ is less than the multiplicity of λ, then A cannot be diagonalized.

j. An eigenvalue for a matrix can never equal 0.

k. An eigenvector for a matrix can never equal **0**.

l. An eigenspace for a matrix can never equal {**0**}.

m. The projection of a vector **b** onto a subspace H is a vector $\mathbf{p} \neq \mathbf{0}$ satisfying two conditions: (1) **p** is an element of H and (2) **b** – **p** is perpendicular to **p**.

n. For any system $A\mathbf{x} = \mathbf{b}$, there is a unique least squares solution $\hat{\mathbf{x}}$.

o. For an inconsistent system $A\mathbf{x} = \mathbf{b}$, the least squares solution $\hat{\mathbf{x}}$ is the closest vector possible to the true solution $\mathbf{x}$.

p. For any $m \times n$ matrix A and vector $\mathbf{b}$ in $\mathbb{R}^m$, the system $A\mathbf{x} = \mathbf{b}$ is always consistent.

q. For any $m \times n$ matrix A and vector $\mathbf{b}$ in $\mathbb{R}^m$, the system $A^TA\mathbf{x} = A^T\mathbf{b}$ is always consistent.

r. For any $m \times n$ matrix A and vector $\mathbf{b}$ in $\mathbb{R}^m$, the system $A^TA\mathbf{x} = A^T\mathbf{b}$ is consistent only if A^TA is invertible .

s. For any $m \times n$ matrix A, A^TA is invertible if and only if A has independent columns.

t. For any $m \times n$ matrix A, A^TA is invertible if and only if A has at least one non-pivot column.

u. If the columns of the matrix A are orthogonal to each other, then A^TA is a diagonal matrix.

v. If the columns of the matrix A are orthogonal to each other, then A^TA is the identity matrix.

w. If $\mathbf{u}$ and $\mathbf{v}$ are nonzero orthogonal vectors, then $||\mathbf{u}+\mathbf{v}|| = ||\mathbf{u}|| + ||\mathbf{v}||$.

x. If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are nonzero vectors in $\mathbb{R}^m$, and if each is orthogonal to the other two, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly independent set.

y. Fitting a parabola as closely as possible to set of data points in the *xy* plane cannot be formulated as a least squares matrix problem because the equation of a parabola is not linear.

z. This list of T/F questions is way too long!