Linear Algebra Class Notesheet: 4/11/2017

1. Example for seagull island: We saw the *k*th population vector is

$$\begin{bmatrix} a_k \\ b_k \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 5 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & -.5 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 3 & -5 \end{bmatrix}\begin{bmatrix} a_0 \\ b_0 \end{bmatrix}.$$

   That simplifies to

$$\frac{1}{8}\begin{bmatrix} 5(1.5^k)a_0 + 3(-.5)^k a_0 + 5(1.5^k)b_0 - 5(-.5)^k b_0 \\ 3(1.5^k)a_0 - 3(-.5)^k a_0 + 3(1.5^k)b_0 + 5(-.5)^k b_0 \end{bmatrix}$$

   and further algebraic rearrangement is possible. This looks complicated, but at least it is an equation for the kth population vector as a function of *k*, and can be used for further analysis.

2. Alternate view.
   a. Let's reconsider the equation $\mathbf{x}_k = PD^kP^{-1}\,\mathbf{x}_0$.
   b. Once we find *P* and *D*, we can proceed to compute $P^{-1}\,\mathbf{x}_0$, which is a particular constant vector. For simplicity call that $\mathbf{y} = [y_1 \;\; y_2 \;\; y_3 \; \ldots \; y_n]^T$.
   c. Then $D^kP^{-1}\,\mathbf{x}_0 = D^k\mathbf{y} = [\;(\lambda_1)^k y_1 \;\; (\lambda_2)^k y_2 \;\; (\lambda_3)^k y_3 \;\; \cdots (\lambda_n)^k y_n\;]^T$.
   d. So $\mathbf{x}_k = P \cdot [\;(\lambda_1)^k y_1 \;\; (\lambda_2)^k y_2 \;\; (\lambda_3)^k y_3 \;\; \cdots (\lambda_n)^k y_n\;]^T$.
   e. Remembering the interpretation of matrix vector multiplication, a linear combination of the columns of the matrix, we find $\mathbf{x}_k = y_1(\lambda_1)^k\mathbf{v}_1 + y_2(\lambda_2)^k\mathbf{v}_2 + \cdots + y_n(\lambda_n)^k\mathbf{v}_n$.
   f. Notice that the only part that changes as we generate the sequence is *k* – the eigenvalues, eigenvectors, and components of **y** are all constant. So we have a linear combination of *n* fixed eigenvectors ($y_j\,\mathbf{v}_j$) each multiplied by powers of their respective eigenvalues.
   g. If there is a unique eigenvalue of greatest absolute value, say λ_1 , then eventually, the powers of all lesser eigenvalues will become insignificant compared to λ_1^k. This shows that as *k* goes to infinity, the sequence will be attracted to the eigenspace of the dominant eigenvalue.

3. Related to An Earlier Idea
   a. Prior section of this outline is related to our earlier approach: representing the initial vector as a linear combination of eigenvectors.
   b. Since $\mathbf{y} = P^{-1}\,\mathbf{x}_0$, we also know

      $\mathbf{x}_0 = P\mathbf{y} = y_1\mathbf{v}_1 + y_2\,\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n$.
   c. When we apply *A* to this *k* times, we find

      $A^k(y_1\mathbf{v}_1 + y_2\,\mathbf{v}_2 + \cdots + y_n\mathbf{v}_n) = y_1A^k\mathbf{v}_1 + y_2\,A^k\mathbf{v}_2 + \cdots + y_nA^k\mathbf{v}_n$.

      But this reduces to $y_1(\lambda_1)^k\mathbf{v}_1 + y_2(\lambda_2)^k\mathbf{v}_2 + \cdots + y_n(\lambda_n)^k\mathbf{v}_n$ because the **v**'s are eigenvectors.

5. Another example: Fibonacci Numbers.
   a. We define the sequence $\mathbf{x}_n = \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$.
   b. Easy to derive $\begin{bmatrix} F_{n+1} \\ F_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix}$, so $\mathbf{x}_{n+1} = A\mathbf{x}_n$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.
   c. Therefore, $\mathbf{x}_n = A^n\mathbf{x}_0$. In more detail, that is $\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
   d. Premultiply both sides of the equation by [1 0] and we find

      $[\,1 \;\; 0\,]\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = [\,1 \;\; 0\,]\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $F_n = [\,1 \;\; 0\,]\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

e. Now we diagonalize the matrix A. The eigenvalues are the roots of the characteristic equation $\lambda^2 - \lambda - 1 = 0$. Using the quadratic formula, the roots are $\frac{1+\sqrt{5}}{2}$ and $\frac{1-\sqrt{5}}{2}$.

f. Using exact algebra, we can find the eigenvectors as $\begin{bmatrix} 1 \\ \frac{1+\sqrt{5}}{2} \end{bmatrix}$ and $\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}$. Thus, we can write $A =$

$$\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{-1}.$$

g. Substituting in the earlier equation, this gives

$$F_n = [\,1 \;\; 0\,] \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

h. Multiplying from the left, we have $[\,1 \;\; 0\,] \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} = [1 \;\; 1]$, and then

$$[\,1 \;\; 0\,] \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} = [1 \;\; 1] \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} = \left[\left(\frac{1+\sqrt{5}}{2}\right)^n \quad \left(\frac{1-\sqrt{5}}{2}\right)^n \right].$$

i. OTOH, to compute $\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we can solve $\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

We apply Cramer's rule, first computing $\det \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix} = \frac{1-\sqrt{5}}{2} - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}-1-\sqrt{5}}{2} = -\sqrt{5}$.

j. Now Cramers rule gives

$$\mathbf{x} = \begin{bmatrix} \frac{\begin{vmatrix} 0 & 1 \\ 1 & \lambda_2 \end{vmatrix}}{-\sqrt{5}} \\ \frac{\begin{vmatrix} 1 & 0 \\ \lambda_1 & 1 \end{vmatrix}}{-\sqrt{5}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{1}{-\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{which is the same as} \quad \begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

k. This is the same as $\begin{bmatrix} 1 & 1 \\ \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Substituting both partial computations in our original equation, we get

$$F_n = \left[\left(\frac{1+\sqrt{5}}{2}\right)^n \quad \left(\frac{1-\sqrt{5}}{2}\right)^n \right] \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right].$$