Elementary Mathematical Models

Supplementary Problems

This document provides additional problems to supplement those in the Elementary Mathematical Models text. These can be used for quizzes, exams, or for additional homework assignments. They can also serve as models for instructors to modify to create their own exercises.

The material is divided into three sections. Section 1 contains problems developed by Dan Kalman. These are further subdivided according to the chapters in the text to which they correspond. The second and third sections contain problems and questions composed by Angela Hare, who taught this material at American University. Exam and quiz questions developed by Dr. Hare are in Section 2; additional reading assignments are in Section 3. These problems and questions are not keyed to particular chapters in the text.

1 Problems by Dan Kalman

Chapter 2: Sequences and Difference Equations

1. A weather station in Wallingford, Connecticut recorded the following temperatures in March of 1990:

Day	March 1	March 2	March 3	March 4	March 5	March 6
Temperature	42	42	34	25	22	34

Use the notation T_n to stand for the temperature n days after March 1, and answer the following questions:

a. What is the numerical value of T_3?

b. What is the numerical value of T_0?

c. If you are told that $T_{33} = 39$, what date does that give you the temperature for?

d. The temperature reading for March 25 was 17. Express this fact using the T_n notation.

2. Given $a_{n+1} = a_n + 2 + 3n$ and $a_0 = 2$ find a_3.

3. There is a pattern in the following number sequence:

$$600 \quad 500 \quad 450 \quad 425 \quad \cdots$$

Each number in the sequence is found by adding 200 to half of the preceding number. For example, add 200 to half of the original 600 to get 500, then add 200 to half of 500 to get 450, and so on. Write a difference equation that expresses this pattern.

4. A biologist observes the way a patch of mold grows in a glass dish, recording the size of the patch (in square inches) each day. She sees that the data follow a pattern: each day the patch is 1.24 times larger than it was the preceding day. On the first day the patch measured 8 square inches. The next day it was $1.24 \times 8 = 9.92$ square inches. The day after that it was $1.24 \times 9.92 = 12.3$ square inches, and so on.

a. Determine how large the patch will be on the fifth day.

b. Write out a difference equation for the pattern.

c. Use a numerical approach to predict when the patch will be 25 square inches.

Chapter 3: Arithmetic Growth

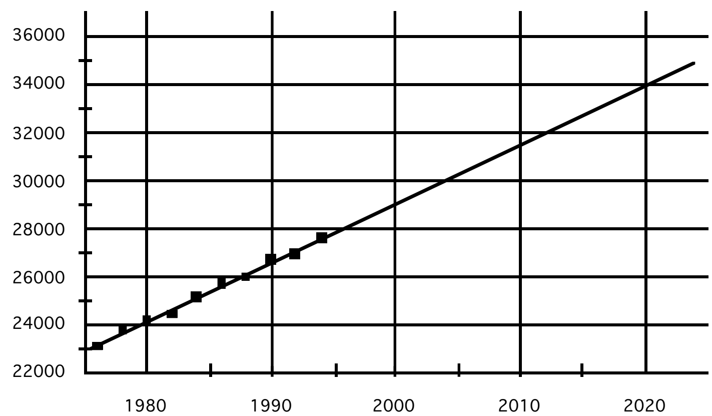
1. A number pattern is given by: $5, 8, 11, 14, 17, \cdots$. Find the 6th and 400th numbers in the pattern.

2. Make up an example of an arithmetic growth model, and write a one page essay about it. Include all of the following in your essay: (1) an explanation of what the model is about, (2) definitions for any variables in your model, (3) a difference equation, (4) the functional equation, (5) a graph for your model, (6) a prediction that uses your functional equation, and (7) a prediction based on inverting your functional equation.

3. Electric Company Study. The electric company in a small town wants to project future growth. They have collected the data in the table below showing how many customers they had in past years.

Year	1976	1978	1980	1982	1984	1986	1988	1990	1992	1994
Customers	23024	23717	24173	24381	25103	25673	25965	26690	26909	27529

This information is shown on the graph below.



The line in the graph was drawn by analysts to estimate future customer needs. The company can presently generate enough electricity to serve a total of 33000 customers.

a. Use a graphical method to estimate when the number of customers will be as high as the company can serve with their current generating capabilities. Write a brief explanation of how you got your answer.

b. The electric company analysts developed the following equation: $C = 22850 + 250n$ where C is the number of customers n years after 1976. Use this equation and either a numerical method or a theoretical method to improve your answer to the preceding question. That is, get a more accurate estimate of when the number of customers will be as high as the company can serve with their current generating capabilities. Show your work or explain how you reached your answer.

4. An economist is studying the cost of housing. In one county, the data show that the average price of a new home increased by about \$2000 per year for each year from 1991 to 1994, starting at

\$185000 in 1991. The economist uses the notation p_n for the average price of a new home n years after 1991. Develop an arithmetic growth model for this problem, as follows: (a) Find a difference equation and a functional equation for p_n, (b) draw a graph for the model (use graph paper), (c) use the model to predict the average home price in the year 2000, and (d) use the model to determine when the average home price will reach \$250000.

5. An ecologist is studying wetlands. In one county, the data show that the number of acres of wetlands has been decreasing by about 200 acres per year for the past 5 years, starting at 36000 acres in 1990. The ecologist uses the notation w_n for the number of acres of wetlands n years after 1990. Develop an arithmetic growth model for this problem, as follows: (a) Find a difference equation and a functional equation for w_n, (b) draw a graph for the model (use graph paper), (c) use the model to predict how many acres of wetlands will remain in the county in the year 2010, and (d) use the model to determine when there will be only 25000 acres of wetlands in the county.

6. Deer Population. The state of Virginia is experiencing a rapid growth in the deer population. One study estimated that there were 5000 deer in Loudon county in 1990, and that the population was growing by 875 deer per year. Develop an arithmetic growth model for this situation, and use your model to answer two questions:

a. What will the deer population be in 1996?

b. When will the population be 13000?

7. Nuclear Reactor. At a nuclear reactor, the cooling system develops a malfunction and is not able to keep the reaction at a constant temperature. The engineers studying the problem estimate that temperature is rising at 28 degrees per hour. The temperature was 1480 degrees at 9 AM. If the temperature reaches 2000 degrees, the reactor will have to be shut down. Develop an arithmetic growth model for this situation, and use your model to answer two questions:

a. What will the temperature be by 4 PM?

b. According to the model, when will the reactor have to be shut down?

You may use any combination of graphical, numerical, and theoretical methods.

8. A police department study showed that in 1995 there were 7200 cases of car theft, while in 1996 there were 8000 cases. Using c_n to represent the number of car thefts for year n, where 1995 is year 0, develop an arithmetic growth model for this situation.

a. What is the difference equation for the model?

b. What is the functional equation for this model?

c. According to the model, how many car thefts will occur in 2007?

d. According to the model, how many years will it take for the number of car thefts to reach 20000?

Chapter 4: Linear Graphs, Functions, and Equations

1. Draw a graph showing a straight line with an x intercept of 5 and a y intercept of 3, and find the equation of that line.

2. When a sealed container is heated, the pressure inside the container rises. This is tested in a laboratory experiment with a sealed air tank. When the tank is at 70 degrees, the pressure inside is 15 pounds per square inch. After heating the tank up to 210 degrees the pressure inside is 20 pounds per square inch. **Find an equation for the temperature and pressure.** You may assume a linear model, meaning that a graph showing temperature on one axis and pressure on the other will be a straight line. **Use your equation to predict what the pressure would be at a temperature of 150 degrees.**

3. A dietician is developing various mixtures of corn and lima beans to provide a predetermined amount of fiber. Let C stand for the amount of corn in an acceptable mixture, and let B stand for the amount of beans. If she uses corn alone, 2.5 cups are required (so $C = 2.5$ when $B = 0$). Similarly, if she uses just lima beans 1.8 cups are needed ($B = 1.8$ when $C = 0$). The dietician decides to use a linear model. That means that a graph of the acceptable mixtures of corn and beans will follow a straight line, when plotted with the amount of corn on one axis and the amount of beans on the other. Using this information, **Find an equation for B and C.** Based on your equation, **How many cups of beans are needed if 1 cup of corn is used?**

Chapter 5: Quadratic Growth Models

1. Find the total $1 + 2 + 3 + 4 + \cdots + 150$

2. Write a one page essay on the subject of quadratic growth models. Tell what a quadratic growth model is, and describe some important aspects of these models, including difference equations, functional equations, and graphs. In addition, give one example of a quadratic growth model. Be as complete as possible.

3. Write a one page essay comparing and contrasting arithmetic growth models and quadratic growth models. Referring to difference equations, functional equations, graphs, and other properties, explain ways in which arithmetic growth models are similar to quadratic growth models, and also explain ways in which these two kinds of model differ. Be as complete as possible.

4. The manager of an amusement park developed a model for predicting the total profits that will be earned by the park for the first few months of 1996. In this model, P_n is equal to the total profit (in dollars) for the first n days of operation in 1996, with $P_0 = 0$. The difference equation in the model is $P_{n+1} = P_n + 25600 + 1200n$

a. Use the difference equation to compute P_1, P_2, and P_3.

b. Is this model an example of arithmetic growth, quadratic growth, or neither? Explain how you can tell.

c. According to the model, the park's profits will total 310000 (dollars) during the first 10 days of the year. Is it reasonable to estimate total profits of about 600000 for the first 20 days of the year? Explain.

d. Give the functional equation for P_n and show or explain how it can be used to find the total profit for the first 90 days of 1996.

5. A drug company is planning to test how several different antibiotics interact. The idea is to use every antibiotic in a test with every other antibiotic. In each test, a patient will be given a combination of the two antibiotics. To provide a comparison, they will also do a test of each drug by itself. The researchers do not want the doctors and patients to know which tests involve two

antibiotics, and which involve just one, because that might influence the results. So they decide to use two kinds of phoney pills (placebos) that contain no medicine. One is red and one is blue, and they will also be tested in combination with antibiotics. The point of this problem is to figure out how many tests need to be run. Each test involves two pills, either two antibiotics, an antibiotic and a phoney red pill, or an antibiotic and a phoney blue pill. Every possible combination of antibiotics will be tested, and each antibiotic will be tested in combination with each of the phoney pills. Let t_n stand for the number of tests that will be needed if there are n different antibiotics.

a. Calculate t_2 and t_3 by counting all the different combinations needed.

b. Suppose you already plan the tests necessary for 12 antibiotics. Then the director of research tells you to include a 13th antibiotic. How many **new** tests will you have to add? Why?

c. Use the answer to the preceding problem to write an equation that relates t_{12} and t_{13}

d. Find a difference equation for t_n.

e. Find a functional equation for t_n.

f. How many test are needed if there are 50 different antibiotics?

6. An electronics store began selling home computers in 1990. In 1990 they sold 25 computers. In the next year they sold 35 computers. The year after that they sold 45 computers. Assuming that the number sold each year follows an arithmetic growth model, the store manager predicts that $25 + 10n$ computers will be sold in year n (where 1990 is year 0). Let T_n be the total sales for the n years starting from 1990. (For example, T_4 is the total sales for 1990 through 1993. T_0 is the total sales over 0 years, so is 0.) The manager came up with the formula $T_n = 5n^2 + 20n$

a. Compute directly from the given information the total sales for 1990 through 1993. Then use the store manager's equation to compute T_4. Are the answers the same?

b. Find a difference equation for T_n

c. Use your knowledge of quadratic growth models to find your own (functional) equation for T_n. Does it agree with the manager's equation?

d. The store manager estimates that the total market for home computer sales is limited to 5000 homes in the store's area. Because of competition with other stores, the store is only expected to be able to sell a total of 1000 computers. According to the manager's equation for total sales over n years, how many years will it take the store to sell this many computers?

Chapter 6: Quadratic Graphs, Functions, and Equations

1. Write a one page essay about the graphs for equations of the form $y = ax^2 + bx + c$, where a, b, and c stand for particular numerical constants. Your essay should name the important features of the graph, and explain how they can be found. You may use an example to illustrate your explanation, but the essay should be a general description that can be used for *any* choice of the constants a, b, and c.

2. A network analysis for a computer system concerns how the number of telephone lines grows with the number of computers on the network. The analysis includes the equation

$$\text{Number of Lines} = 1 + .5n + 1.5n^2$$

where n is the number of computers on the network.

a. According to the equation how many phone lines would be needed for a total of 20 computers on the network?

b. The network manager's budget can only afford 3000 phone lines. How many computers can be included in the network?

c. In this model, if the number of computers is doubled, what is the approximate effect on the number of telephone lines that will be needed?

3. One problem in the book concerned a factory making backpacks. A model was developed for the way profits from the factory depend on the price that is charged for the backpacks. According to the model,

$$\text{PROFIT} = -.3p^2 + 25.2p - 434.3$$

where p is the price charged for each backpack (in dollars) and the profit is given in thousands of dollars. Using the equation answer the questions below. [Hint: Imagine making a graph of the model with the price p on the x axis, and with the profit on the y axis. Then the equation above would be $y = -.3x^2 + 25.2x - 434.3$. How would that graph look?]

[answer:] This is a quadratic equation, and the graph will be a parabola. Since the coefficient of x^2 is a negative number, we know it will open downward ($\cap$).

a. How much profit would the factory make by selling the backpacks for \$30 each?

Answer: The price is given as 30. Set $p = 30$ in the equation

$$\text{PROFIT} = -.3p^2 + 25.2p - 434.3$$

and compute

$$\text{PROFIT} = -.3(30)^2 + 25.2(30) - 434.3 = 51.7$$

This is in units of thousands, so the profit will be \$51700.

b. In order to undercut the competition, the factory owners decide to sell the backpacks at a price that will produce no profit at all. They want to make the price as low as possible, without actually losing any money. Find the price they should set to obtain a profit of zero.

Answer: This time we are given the Profit (0) and want to find the price. The equation is

$$0 = -.3p^2 + 25.2p - 434.3$$

or, in a more familiar form, reverse the equation and use the variable x in place of p :

$$-.3x^2 + 25.2x - 434.3 = 0$$

This is a quadratic equation, with $a = -.3$, $b = 25.2$ and $c = -434.3$. The solutions are given by the quadratic formula as

$$x = \frac{-25.2 \pm \sqrt{25.2^2 - 4(-.3)(-434.3)}}{-.6}$$

Working through the calculations, that gives two values for x : approximately 24.21 and 59.79. The owners want to choose as low a price as possible, so the correct answer would be \$24.21 per back pack. If you substitute that price for p in the profit equation, you find a result of

$$\text{PROFIT} = -.3(24.21)^2 + 25.2(24.21) - 434.3 = -0.0452$$

Remember this is a profit in thousands of dollars, so actually works out to a \$45.20 loss. If you use more decimal places in the answer, say \$24.214238, then the profit will come out much closer to 0, to about one tenth of a cent, in fact. This verifies that the quadratic formula gives the correct theoretical answer, although no one would actually set a price correct to 6 decimal places!

c. What price should be charged to make the highest possible profit?

Answer: The highest profit corresponds to the highest point on the graph, again because the profits are shown on the vertical axis. That is the vertex. We know from our study of quadratics that the vertex is located according to the equation $x = -b/2a$ which in this case is $x = (-25.2)/(-.6) = 42$. When the price is set at \$42, the profit will be at the highest point of the graph. That gives the maximum possible profit, according to the model.

4. In a market analysis, a frozen yogurt chain develops a model for profits as a function of unit price. They find the equation

$$P = -218.5u^2 + 252.4u - 1.38$$

where P is the daily profit (in hundreds of dollars) from the stores in the chain, and u is the basic unit price (in cents) for a cup of yogurt. Based on this model, determine what price the chain should set to get the largest possible profits. Also, determine the range of prices they can select without losing money. That is, the range of prices for which the profit is 0 or more.

Chapter 7: Polynomial and Rational Functions

1. Write a one page essay about the graphs of polynomial and rational functions. In your essay, describe the kinds of shapes one should expect in the graph of a polynomial. What does the equation of the polynomial tell you about the shape? Similarly, describe the kinds of shapes that can be expected in the graphs of rational functions, and how important features of a graph can be determined from the equation of a rational function. Be as complete as possible.

2. Answer the questions below about the graph of the equation

$$y = \frac{(3x-6)(x+2)}{(x+1)(x-3)} \quad \text{which can also be written} \quad y = \frac{3x^2-12}{x^2-2x-3}$$

a. Would the graph have any vertical asymptotes? If so, tell where, and how you know.

b. Would the graph have any horizontal asymptotes? If so, tell where, and how you know.

c. Would the graph have any x intercepts? If so, tell where and how you know.

d. Would the graph have any y intercepts? If so, tell where and how you know.

3. The figure below shows a graph. Determine which equation below best matches the graph, by

eliminating all the rest. For each equation that you eliminate, tell your reason.

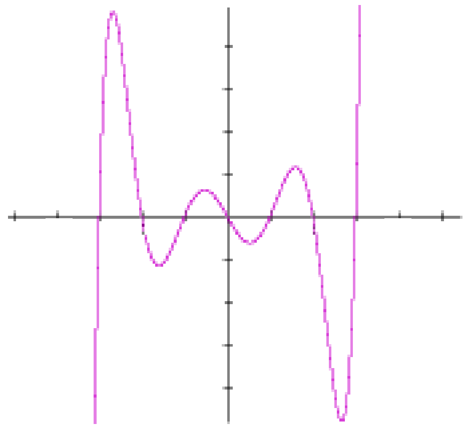
a. $y = x^5 - 50x^3 + 300x$

b. $y = x^7 - 35x^5 + 300x^3 - 350x + 100$

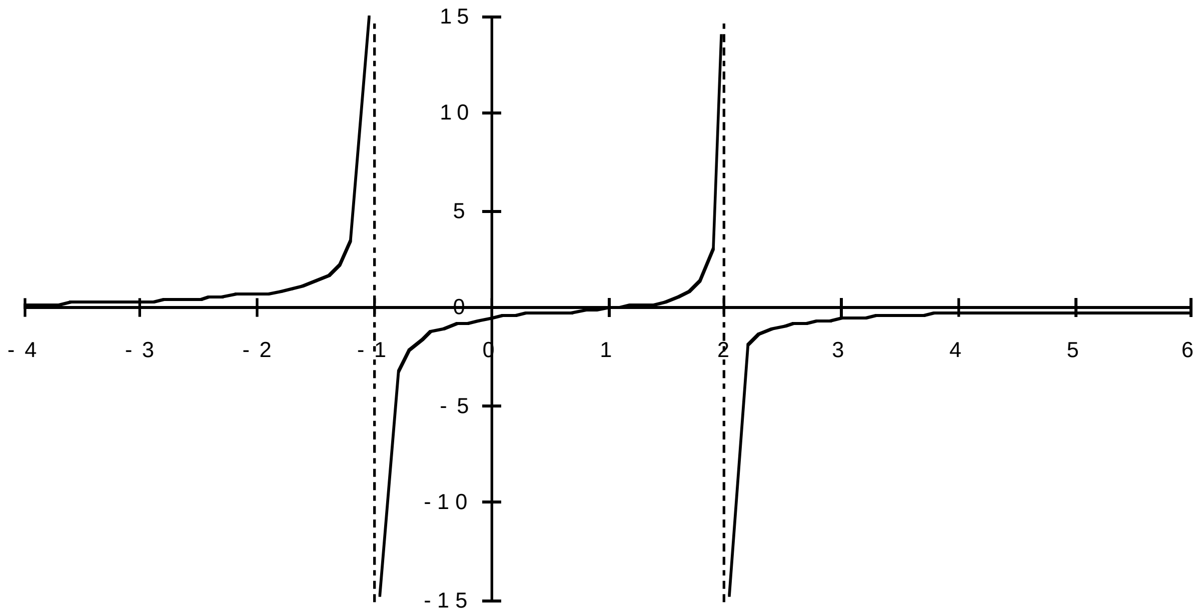
c. $y = -x^7 + 35x^5 - 300x^3 + 350x$

d. $y = x^8 - 85x^5 + 300x^3 - 150x$

e. $y = x^9 - 85x^5 + 300x^3 - 150x$



4. Consider the figure below:



a. Does the figure show the graph of a polynomial function? Of a rational function? Justify your answer.

b. Which of these equations best matches the graph above?

$$y = (x+1)(x-2)(1-x) \qquad y = \frac{(x+1)(x-2)}{(1-x)} \qquad y = \frac{(1-x)}{(x+1)(x-2)}$$

Explain your reasoning.

Chapter 8: Fitting a Line to Data

1. This problem concerns fitting a straight line to the following data points: (1,5.2), (2,6.9), (3,8.9), (4,11.1).

a. One possible line has the equation $y = 2x + 3$. Using that line, compute the error for the 3rd data point: (3,8.9). Show all your steps.

b. The total error function for this problem is given by Total Error $= 30m^2 + 20mb + 4b^2 - 180.2m - 64.2b + 277.07$. Use this to compute the total error for the line $y = 2x + 3$. Explain your steps.

c. Change the b in the equation for total error to 3. Now you have an equation for the total error for any line with $b = 3$. Use this equation to find the choice of m that results in the smallest possible total error (based on the fact that $b = 3$).

Chapter 9: Geometric Growth

1. There were 20000 students enrolled in a school district in 1990. Over the next several years, enrollments were seen to increase by roughly 8 percent per year. Assuming that the enrollments continue to increase by 8 percent every year, develop a difference equation model for the enrollment figures. Find the functional equation for your model, and use it to predict the number of students in the school district for the year 2005.

2. A researcher is monitoring the way pollutants from a chemical spill dissipate over time. In her initial measurement, on August 1, 1995, she found about 20 parts per million of pollution in a soil sample. She continued making measurements once a month, and found that the amount of pollution decreased by 28% with each measurement. Assuming that the pollution continues to decrease by 28% every month, develop a difference equation model for the pollution data. Find the functional equation for your model, and use it to predict the level of pollution (in parts per million) by January 1, 1998.

3. A continuous geometric growth model describes the way food is chilled after being placed in a freezer at 0 degrees (Fahrenheit). The model includes the equation:

$$T = 70(.78^{t/30})$$

where T is the temperature of the food and t indicates how long the food had been in the freezer: T is given degrees Fahrenheit and t is given in units of minutes.

a. There are three numbers that appear in the equation: 70, .78, and 30. Tell what each of these numbers tells you about the model.

b. According to the model, what will the temperature of the food be 2 hours after it has been placed in the freezer?

c. When does the food reach a temperature of 10 degrees? Give an answer in minutes correct to 2 decimal places.

4. (This problem reconsiders the police department study discussed in a problem for Chapter 2.) A police department study showed that in 1995 there were 7200 cases of car theft, while in 1996 there were 8000 cases. Using c_n to represent the number of car thefts for year n, where 1995 is year 0, develop a geometric growth model for this situation.

a. What is the difference equation for the model? [**Hint:** What is the growth factor from 7200 to 8000?]

b. What is the functional equation for this model?

c. According to the model, how many car thefts will occur in 2007?

d. According to the model, how many years will it take for the number of car thefts to reach 20000?

5. One very simple model of inflation uses geometric growth to describe how the price of some commodity increases over time. For example, suppose a gallon of milk costs \$1.45 today, and that the price increases by 2.8% every 6 months. Assuming a geometric growth model, what is the equation that tells the price of a gallon of milk t years from now?

6. A continuous growth model describes how impurities are removed from a chemical solution in a manufacturing process. The model includes the equation

$$A(t) = 100 \cdot .36^{t/4}$$

where t is time in hours from the start of the process, and $A(t)$ is the amount of impurities (in grams) after t hours.

a. Fill in the boxes: At the start of the process there are $\boxed{}$ grams of impurities, and that amount is multiplied by a growth factor of $\boxed{}$ every $\boxed{}$ hours.

b. Use the equation $A(t) = 100 \cdot .36^{t/4}$ to determine how long it takes for the impurities to be reduced to 5 grams. Use an algebraic method, and give your answer correct to 6 decimal places

Chapter 10: Exponential Functions

1. About e. Complete the paragraph below about e. You must circle either CAN or CANNOT in two places and fill in one blank. Then, at the end, write a short statement describing one of the special properties of e.

The number e is found in many books and articles involving exponential functions. Functions of the form $10^{t/6}$, $2^{4.1t}$, and 1.46^t (CAN / CANNOT) always be written using e as a base. The numerical value of e is approximately ________________________. The exact value of e (CAN / CANNOT) be given if enough decimal places are used. There are many interesting properties of e, and e just sort of pops up naturally in many problems connected with exponents. One of the special things about e is the following: (Write a short explanation.)

2. Logarithms

a. Explain what a logarithm is. For example, according to the calculator, $\log 2 = .30103$. Explain what this means, and how you could check that it is correct. **[This is an essay question.]**

b. Explain what a natural logarithm (ln) is. For example, according to the calculator $\ln 10 = 2.3026$. Explain what this means, and how you could check that it is correct. [**This is an essay question.**]

c. Use logarithms to find the solution to the equation $10^x = 4.5$

d. Use logarithms to find the solution to the equation $1.1^x = 3$

Chapter 11: More On Logarithms

1. Logarithmic Scales.

a. Explain what a logarithmic scale is and why it is used. [**This is an essay question.**]

b. The pH scale is used to measure acidity. The pH of a liquid is the negative of the log of the concentration (in moles per liter) of hydrogen ions. In apple juice, there are .000794 moles per liter of hydrogen ions. What is the pH of apple juice?

c. The Richter scale is a logarithmic scale used to measure the strength of an earthquake. Compare two earthquakes, one measuring 4 on the Richter Scale, and the other measuring 8 on the Richter Scale. Which quake is more powerful? How many times more powerful is it?

Chapter 12: Geometric Sums and Mixed Models

1. For the parts of this problem, use the initial value $a_0 = 100$ and the difference equation $a_{n+1} = .6a_n + 5$.

a. Find a_1, a_2, and a_3

b. Find the functional equation for this problem.

c. Find a_{25} using the functional equation. [**NOTE:** if you are not sure your answer to part b is correct, use this equation: $a_n = 100(.6)^n + 12.5(1 - .6^n)$]

d. Does this model eventually level off to a steady number? If so, what is the steady number? How do you know?

2. A doctor is studying the way a medication is eliminated from the body. She finds that every 4 hours, one fifth of the drug is eliminated. She starts the patient off with 1000 units of the medication, followed by 250 units every four hours. Let a_n be the amount of drug in the blood just after the n^{th} 250 unit dose is taken. So for example, a_3 is the amount of drug in the blood right after the 3^{rd} 250 unit dose is taken.

a. Find the numerical values of a_1, a_2, and a_3

b. What is the difference equation for a_n ?

3. I owe \$5000 on my credit card. Each month, the card company adds a finance charge of 1.5% of the balance. I plan to pay 100 dollars per month, and not make any more purchases. This situation can be analyzed using difference equations, as follows. I will use b_n to stand for the amount I owe after making n payments, with $b_0 = \$5000$. The difference equation is $b_{n+1} = 1.015b_n - 100$.

a. What will by balance be after 2 years (that is, after 24 payments)? [**HINT:** There is a faster method than using the difference equation 24 times.]

b. How long will take to pay off everything I owe?

4. A finite math student is exploring a difference equation model. She obtains the following pattern of results:

$$\begin{aligned} c_0 &= .4 + 25 \\ c_1 &= .4(1.6) + 25 \\ c_2 &= .4(1.6)^2 + 25 \\ c_3 &= .4(1.6)^3 + 25 \\ c_4 &= .4(1.6)^4 + 25 \end{aligned}$$

Based on this pattern answer the following questions:

a. What is c_8? What is c_n?

b. What do you get if you add up c_0 through c_4?

c. What do you get if you add up c_0 through c_8?

d. What do you get if you add up c_0 through c_n?

e. Use an appropriate shortcut to with these patterns so that you can give a functional equation for the total of the data values c_0 through c_n.

5. A drug company is testing a pain reliever. The researchers know that in 6 hours the body removes about 4/5 of the drug. In one experiment, each patient is given an initial dose of 200 units of the drug, followed by repeated doses of 100 units every 6 hours. That leads to the difference equation $d_{n+1} = (1/5)d_n + 100$, where d_0 is the amount of drug in the body immediately after taking the initial dose, and for $n \geq 1$, d_n is the amount of drug in the body immediately after taking the n^{th} dose of 100 units.

a. Find a functional equation for d_n

b. Use your functional equation for d_n to compute d_6

c. According to the difference equation, if $d_n = 125$, what will d_{n+1} equal? What does this show?

d. A doctor would like the amount of drug to level off at 150 units for one patient. This can be achieved by changing the amount of drug that the patient takes every 6 hours from 100 units to something else. What should that repeated dosage amount be? Show how you got your answer, and check that it gives the correct results.

6. A large company gives its workers a raise every year. The raise is computed as 6 percent of the previous year's salary, plus $500. Suppose a new employee starts out at $25000. Let s_n be the salary in year n, where the first year of employment is considered to be year 0. Develop a difference equation for s_n. What kind of model is this?

7. A pollution study developed a model for the amount of PCB's in a large lake. In the study, p_n is the amount of PCB's (in parts per million) at the start of month n where the study is considered to have started at the beginning of month 0. The model used the difference equation

$$p_{n+1} = .82p_n + .005; \quad p_0 = .02$$

Find a functional equation for this model. According to the model, what prediction can be made for the future levels of PCB pollution in the lake?

Chapter 13: Logistic Growth

1. For all of the parts of this problem, use the difference equation $a_{n+1} = .025(100 - a_n)a_n$.

   a. If $a_0 = 10$, find a_1, a_2, a_3, and a_4

   b. Predict approximate values a_n for large n, say for a_{50} or a_{100}

   c. Find a fixed point for this model. That is, find a number for a_n that leads to exactly the same number for a_{n+1}

2. A science lab is investigating several different models for the way weeds grow in a lake. In each model, a_n is the number of square feet covered by weeds after n weeks. In each part below, a difference equation is given for one of the models. For each one, tell what the long range predictions of the model will be, and whether you think these predictions make sense.

   a. $a_{n+1} = .004(1300 - a_n)a_n$.

   b. $a_{n+1} = .002(1300 - a_n)a_n$

   c. $a_{n+1} = .0007(1300 - a_n)a_n$

3. A logistic growth model is being developed to describe how bacteria grow in unrefrigerated hamburger. For a one pound sample of hamburger, it is found that the bacteria increase each hour with a growth factor r given by the equation $r = .0004(6000 - p)$ where p is the number of bacteria at the start of the hour.

   a. At the start of the experiment, there are 1000 bacteria in the hamburger. What is the growth factor r at that point, and how many bacteria will there be after an hour?

   b. Later in the same experiment it is found that there are 3200 bacteria. What is the growth factor r at that point, and how many bacteria will there be after another hour?

   c. What is the difference equation for this model?

4. (This is an extended problem that would be suitable for a worksheet or a homework assignment.)

This problem will give you some practice working with the ideas of logistic growth.

A laboratory is developing a procedure to grow a certain kind of mold that will be used to make a new antibiotic. The mold is grown in a vat with a nutrient solution made up of sugars, water, and other ingredients. For each different nutrient solution, the laboratory makes two tests by introducing a known amount of the mold and observing how the mold grows over a 24 hour period. For each of the two test amounts, the growth factor for the 24 hour period is recorded. The results of these tests are shown in the table below.

Test	First Test		Second Test	
Solution	Population	Growth Factor	Population	Growth Factor
A	100	0.8	500	0.6
B	100	0.9	400	0.3
C	100	2.0	500	1.2
D	100	3.0	600	1.5
E	100	4.0	600	1.5

Make a separate analysis for each of the five nutrient solutions. For each one, your job is to develop and analyze a logistic growth model. In doing this work, follow the outline below:

a. Develop a linear equation that relates the growth factor to the population size. Using the given data, graph a line showing the growth factor on the vertical axis and the population on the horizontal axis. From the graph, and work out the equation for r and p.

b. Formulate a difference equation in the form $p_{n+1} = m(L - p_n)p_n$. What are the constants m and L for this model?

c. Make a graph using p_n as the x value and p_{n+1} as the y value. Find the highest point on the graph. Use this to decide whether the model will always produce values of p_{n+1} that are between 0 and L for your L. Relate your conclusion to the condition $mL < 4$.

d. Is there a value of the population that results in a growth factor equal to 1? What does that tell you about the future population growth for this model? Relate your conclusion to the value of $L - 1/m$.

e. Verify your conclusions by computing and graphing the first several values of population model for some starting population.

5. In the preceding exercise, you were supposed to develop logistic growth models for growing mold in a laboratory under a variety of conditions. The difference equations that you should have found are shown below.

A. $p_{n+1} = 0.0005(1700 - p_n)p_n$
B. $p_{n+1} = 0.002(550 - p_n)p_n$
C. $p_{n+1} = 0.002(1100 - p_n)p_n$
D. $p_{n+1} = 0.003(1100 - p_n)p_n$
E. $p_{n+1} = 0.005(900 - p_n)p_n$

a. For each difference equation, use the results of the logistic growth chapter to decide if the populations eventually dies out, levels off, remains valid without leveling off, or eventually leads to negative (and so invalid) predicted population sizes.

b. For model C. above, investigate the effect of harvesting various amounts from the model. For example, if you harvest 200 members of the population from each cycle, the difference equation will become

$$p_{n+1} = 0.002(1100 - p_n)p_n - 200$$

Describe what will happen to this population over the long term in terms of the starting population.

c. Repeat the preceding problem using a harvest of 400, then of 500.

6. A scientist is studying a certain type of plant virus and how it spreads through fields of wheat. In one experiment, there were initially 100 infected plants, and in one week that number had doubled to 200 infected plants (a growth factor of 2). In another experiment the scientist started with 600 infected plants, and after a week that number had grown to 900 infected plants (a growth factor of 1.5.) Based on this information develop a linear equation in which the variables are r, the growth factor for infected plants over a week, and p, the number of infected plants at the start of that week. Then use this equation to develop a logistic growth model for the number of infected plants week by week as an infection spreads.

7. A chemist is studying the way a crystal forms in a solution, and develops a logistic growth model. The following equation is used:

$$\text{Growth Factor} = 2 - .0002 \cdot (\text{Amount of Crystal})$$

where the amount of crystal is given in grams and the growth factor is for one hour. For example, when there are 800 grams of crystal, there is a growth factor of $2 - .0002 \cdot 800 = 1.84$. That means after one hour the crystal will grow to $800 \cdot 1.84 = 1472$ grams. Use the equation for Growth Factor to develop a logistic growth model for the crystal. Let c_n be the amount of crystal after n hours, starting with $c_0 = 100$. In your answer, include

a. The first few terms: c_1, c_2, and c_3.

b. A difference equation for c_n

c. A discussion of what the model predicts for the future growth of the crystal.

8. At the start of the summer, algae starts to grow in a pond. An ecologist finds that the at first the algae spreads rapidly, but over time the algae spreads more and more slowly. An equation for this situation is given by $A_{n+1} = .04(60 - A_n)A_n$ where A_n is the number of acres covered by algae at the end of the n^{th} week of the study.

a. What kind of a model is this?

b. What kind of long term predictions does this model lead to? How can you tell?

c. Assuming that at the start of the study the algae covers 10 acres, show that your prediction from part c is correct.

d. If the starting amount of algae is different, say 5 acres, or 20 acres, how does that affect the long term predictions. Explain.

Chapter 14: Chaos in Logistic Models

1. Write a one page essay on the subject of Chaos. Include the following in your answer:

- A comparison for chaotic versus nonchaotic models of how small errors in parameters or initial values affect the predictions of the model;
- A description of the so called *butterfly effect*;
- An explanation of linear versus non-linear difference equations, and the significance of this distinction to the concept of chaos.

Add any other important aspects of chaos. Be as complete as possible.

2. Write a one page essay about chaos in logistic models as discussed in Chapter 14. Include the following in your answer:

- A discussion of the overall idea of chaos;
- A description of the specific form that chaos takes for logistic models;
- An explanation of the effects of small changes in the initial population size or in the parameters of a chaotic logistic growth model.
- A comparison of the feasibility of making long term predictions in logistic growth models that are chaotic and in models that are chaotic

Add anything else that is significant about chaos in logistic models. Be as complete as possible.

Review Problems for Growth Models

These problems require the students to apply their knowlwdge of several different kinds of growth models. The problems would be suitable for review at the end of the course. They can also be used on an exam that covers several different kinds of models. In particular, these are appropriate problems for a final exam.

1. A research team is studying the outbreak of mad cow disease in England. They estimate the number of new cases of the disease each week as follows:

Week Number	0	1	2	3
New Cases	1600	2000	2500	3125

a. What kind of growth model would be appropriate for this problem? Why?

b. Develop an appropriate model for the data. Define the variables that you use, and give both a difference equation and a functional equation, clearly labeled.

c. According to your model, how many new cases should be expected in the fourth and fifth weeks?

d. According to your model, how many cases should you expect to see in week 20?

2. In each part below a sequence of numbers is given. For each sequence decide whether it is an example of (A) arithmetic growth, (B) quadratic growth, (C) geometric growth, or (D) not any of those. For full credit **YOU MUST TELL HOW YOU REACHED EACH ANSWER.**

a. 11.0, 11.5, 12.5, 14.0, 16.0, 18.5

b. 25.0, 30.0, 36.0, 43.2, 51.84

c. 25.6, 28, 31, 34.75, 39.44, 45.3

d. 5.0, 6.7, 8.4, 10.1, 11.8

3. An investment company installed an email system for its employees to use. This system used a central computer to store and deliver messages. The amount of time employees spent using the email system was recorded each month (in units of minutes), producing the following data: 317.11, 417.11, 538.54, 681.38, 845.64, 1031.31, 1238.40. Would this data be most accurately modeled as arithmetic growth, quadratic growth, or geometric growth? How can you tell?

4. For each situation below, indicate whether the problem should be modeled using arithmetic growth, quadratic growth, geometric growth, mixed growth, logistic growth, or none of these.

a. A population is growing by increasing amounts each year. The first year it grows by 1000, the next year it increases by 1300, the year after that the increase is 1600, then 1900, then 2200, then 2500, and so on.

b. A biologist is working on a biological filter for purifying water. Each time the water is passed through the filter, 87 percent of the inpurities are removed. That is, when you pass the water through the filter the first time, 87 percent of the impurities are removed. Then, if you pass the water through a second time, 87 percent of the remaining impurities are removed, and so on. The model should describe how the total amount of impurities shrinks as the water is put throught the filter over and over again.

c. A financial planner is managing a trust fund for a college student. The trust fund grows by about 9 percent each year as a result of interest and dividends. The manager pays out $25000 per year to the student for tuition and living expenses. The model should describe how the total amount in the fund goes up or down year by year.

5. You are given the following sequence of data values:

4.560 4.013 3.531 3.108 2.735 2.407

a. What kind of growth do you observe in the data values, *arithmetic* growth, *quadratic* growth, or *geometric* growth? Justify your answer.

b. Find a difference equation **OR** a functional equation that gives a good approximation to the data. (Tell whether your answer is a difference or a functional equation.)

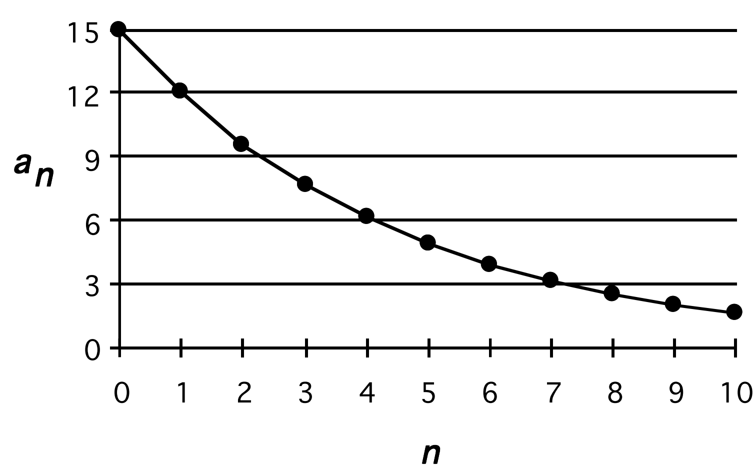
6. Beside the graph below are several difference equations. Pick the difference equation that best matches the graph, and explain why your choice is best.

a. $a_{n+1} = a_n - 3$

b. $a_{n+1} = 1.8a_n$

c. $a_{n+1} = .8a_n$

d. $a_{n+1} = -1.8a_n$



2 Sample Exam Questions by Angela Hare

The following is a collection of problems and questions I used for exams. Each exam consisted of approximately six questions; the final exam was cumulative and a bit longer. During the second half of the course, I included a cover sheet which included formulas such as the quadratic formula, the sum of a geometric series, and some of the more elaborate difference and functional equations.

Problem 1

Consider the following information from an I.R.S. "Tax Due" chart:

Income	Tax Due
3000	454
4000	604
5000	754
6000	904
7000	1054
8000	1204
9000	1354
10000	1504
11000	1654
12000	1804

Let t_n be the tax due on an income of $1000(n+3)$ dollars. Then t_0 is the income on $1000(0+3) = 3000$ dollars, or \$454.

a. Does the amount of tax due follow an arithmetic growth pattern? Explain your answer in a sentence.

b. Find t_1 and t_5.

c. Write a difference equation for t_n.

d. Write a functional equation for t_n.

e. At what income do you pay more than \$5000 tax?

Problem 2

Consider the difference equation $c_{n+1} = 1.08c_n + 50$ with $c_0 = 100$.

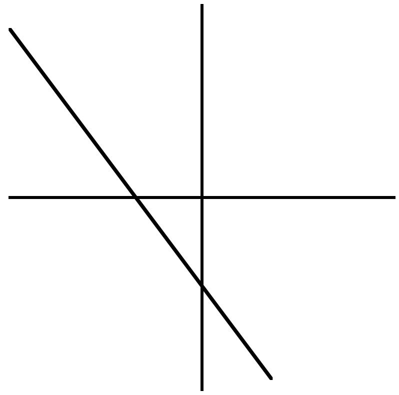
Find the value of c_1, c_2, and c_3.

Problem 3

Write a one paragraph answer to the question "What exactly is the slope of a line?"

Problem 4

Looking at the graph below, a student calculated the slope to be 1.333



Just from looking at the graph, explain how you know that this calculation is incorrect?

Problem 5

Given that $a_{n+1} = a_n + 11$ and $a_0 = 30$, find a functional equation for a_n. Use your equation to find a_{42}.

Problem 6

There is a pattern in the following number sequence. Write a difference equation which expresses this pattern.

8192 2048 512 128 32 8 2

Problem 7

An electronics store began selling home computers in 1990, when they sold 25 computers. Each year after 1990, their sales of computers increased by 20 computers. Write an equation expressing sales of computers (C) as a function of the year (T). Use this equation to predict the year in which sales will surpass 200 computers.

Problem 8

A college student invests $1000 in an account which pays 8% interest, compounded annually. This means that at the end of each year, the interest is computed by finding 8% of the amount currently in the account. For example, after the money has been invested for one year, $1000*0.08=$80 is added to the account, bringing the balance to $1080. At the end of the second year, $1080*0.08=$86.40 is added, bringing the balance to $1166.40. The balance in the account for the first five years is shown in the table below.

Balance

initially	\$1000.00
after 1 year	\$1080.00
after 2 years	\$1166.40
after 3 years	\$1259.71
after 4 years	\$1360.49
after 5 years	\$1469.33

a. Does the growth of the money in the account illustrate arithmetic growth? Explain why or why not.

b. Let b_n represent the amount of money in the account n years after 1990. Write a difference equation for b_n.

Problem 9

[For this problem, each student had a code number given as the first three digits of his/her student number, with 0 replaced by 10.]

Consider the quadratic equation $y = ax^2 + bx + c$, with the coefficients a, b, and c equal to the three digits of your code number. For instance, if your code is 147, then your equation will be $y = x^2 + 4x + 7$). Write your equation below:

$$y = \underline{\hspace{2cm}}x^2 + \underline{\hspace{2cm}}x + \underline{\hspace{2cm}}$$

Now answer the following questions about your quadratic equation:

a. List two ordered pairs that are found on the graph of your equation. For each ordered pair, show work or an explanation which explains why the pair you choose is on the graph.

b. The graph of your quadratic equation may have 0, 1, or 2 x–intercepts. State the number of x–intercepts your parabola has, and explain how you know this. Note: it is not necessary that you find the value(s) of the x–intercepts.

Problem 10

According to a 1993 news report, the population of sub-Saharan Africa is growing at a rate of 2% per year. Assume that 500,000 people lived in a particular region of sub-Saharan Africa in 1993 and that this region also grows at a rate of 2% per year.

a. What do you predict for the population of this region in 1996?

b. Let p_n represent the population of this region n years after 1993. Write a difference equation and functional equation for p_n.

c. The news article claims that the population of sub-Saharan Africa will double in 20 years. Is it true that the population of the region discussed in parts (a) and (b) will double in 20 years? Give reasons to support your answer.

Problem 11

A manufacturer produces steel rings for use in industry, which cost 12 dollars each to produce. There are also fixed costs of \$4000 per month to operate the factory. At a price of 20 dollars each, the manufacturer sold 10,000 rings in a month, and after raising the price to 25 dollars each, sold 8,000 per month. Using a linear demand model, the manufacture came up with the following equation for sales (S) as a function of price (p):

$$S = -400p + 18000$$

Using this equation and the definition of revenue, we can find the revenue equation, which gives revenue (R) as a function of price (p).

$$R = p(-400p + 18000) = -400p2 + 18000p$$

According to the information at the beginning of the problem, the cost to produce S rings is $C = 12S + 4000$. Plugging in our equation for sales, we get the following equation for Cost as a function of price:

$$C(p) = 12(-400p + 18000) + 4000$$

or

$$C(p) = -4800p + 220000$$

a. Assume the manufacturer chooses to sell the rings for \$20 each. Find the sales, cost, revenue, and profit for this price: (Remember to show your work for each part)

b. Write an equation for profit as a function of price, and explain how you found this equation.

c. What price should the manufacturer charge to receive maximum profit, assuming the models above are correct? Be sure to explain how you determined your answer.

Problem 12

On the planet of Perelandra, the children are taught to be extremely respectful of their teachers. When children begin school, they learn that when they enter a classroom, each child must shake hands (or what we would call hands) with the teacher two times during the class - once when they enter the class and once when they leave. They are also taught to respect their peers, and at the beginning of class, each child shakes hands with every other student in the class, in addition to shaking the teacher's hand. They do not shake hands with their peers when they leave class. A sociologist from earth is studying this culture, and she is intrigued by this practice. It seems to her that there will be an awful lot of handshakes involved in large classes.

a. Let h_n represent the number of handshakes necessary in a class of n students. Using this definition, find h_0, h_1, h_2, and h_3.

b. Write a difference equation and functional equation for h_n.

c. How many handshakes take place in a class of 100 students?

Problem 13

For this problem, c_1 and c_2 are the first two numbers of your code.

A young couple has their first child, and they want to begin saving for her college education. They begin by investing $(c_1 \cdot 1000)$ dollars in an account which pays $c_2\%$ interest compounded monthly. (So if your code is 147, they invested \$1000 in an account paying 4% interest, compounded monthly). Let b_n represent the balance they have in the account after n years.

a. Write a difference equation and functional equation for b_n.

b. How much will they have in the account after 18 years, when their daughter is ready for college?

c. If they hope to have \$25,000 in the account after 18 years, how much should they invest initially?

Problem 14

Explain in several sentences the differences between quadratic growth and geometric growth. How can you recognize each type? Include in your answer an example of a situation which is described well by quadratic growth, as well as an example of a situation described by geometric growth.

Problem 15

Using the variables x and y, write the equation of a line which goes through the points (0,2) and (4,6). Graph this line on a coordinate plane.

Problem 16

Solve for x: $3.7^x = 42.1$

Problem 17

Solve for x: $2000 = 5(1.018)^x$

Problem 18

Solve for x:

$$0 = 2000(1.005)^{18} - x\left(\frac{1.005^{18} - 1}{1.005 - 1}\right)$$

Problem 19

Is $\log_5 0.01$ a positive or negative number? Give a reason for your answer.

Problem 20

What is the sum $1 + 1.5 + 1.5^2 + 1.5^3 + \cdots + 1.5^{11}$?

Problem 21

Imagine that you graduate from college and after a few years plan to purchase your first new home. You need to get a loan or mortgage of $150,000. The bank charges an APR (annual percentage rate) of 8.5% for a 20-year mortgage, paid in monthly installments. Let b_n stand for the balance you still owe after n months.

a. What is b_0?

b. Write a difference equation for b_n. (You will have to use a variable, such as d, to represent the monthly payment, since it is not known.)

c. Write a functional equation for b_n. Again, you will need to leave the monthly payment as a variable.

d. Use the functional equation you have written to determine what your monthly payment will be, assuming that you pay off the mortgage in exactly 20 years, or 240 months.

e. If you make the monthly payment you found in part (d) for 240 months, what is the total amount you will pay to the mortgage company?

f. (Extra credit) Instead of paying the mortgage in 20 years, what if you were able to make $1500 payments every month until the loan is paid off? How many months will it take for you to pay off the mortgage?

Problem 22

A doctor wants to give her patient a new medication, and she would like for the medication to eventually level off in the body to 100 milligrams. The patient takes a dose every 3 hours, and half of the medication is removed from the body every three hours.

a. What dose should the doctor prescribe for the patient to take every three hours?

b. Let d_n represent the amount (in mg's) of medication left in the body after n three-hour periods. Write a difference equation and functional equation for d_n.

c. How many mg of medication are in the patient's body after 12 hours? Has the medication leveled off yet?

d. If the doctor wants to increase the medication so that it levels off to 300 mg in the body eventually, what should the repeated dosage be?

Problem 23

Assume a young man named Harry is purchasing a $40,000 car. He takes out a loan for that amount, which charges an 8% APR (annual percentage rate), with monthly payments, and Harry assumes he can pay off the loan eventually by making $250 payments each month. Remembering his study of mixed models in Elementary Math Models in college, Harry sets up the following functional equation, in which b_n represents the amount he still owes after n months.

$$b_n = 40000(1.0067)^n - 250\left(\frac{1.0067^n - 1}{1.0067 - 1}\right)$$

Harry wants to know how long it will take for him to pay off the loan, so he sets b_n to 0 and tries to solve the following equation for n:

$$0 = 40000(1.0067)^n - 250\left(\frac{1.0067^n - 1}{1.0067 - 1}\right)$$

After simplifying the equation as much as possible, he ends up with the equation

$$-13.89 = 1.0067^n$$

When he tries to solve for n by taking $(\log -13.89)/(\log 1.0067)$, he gets an error message on his calculator. What is wrong with Harry's assumptions? (You can assume that Harry is not making any arithmetic errors, and he is using the correct equations). What must he do in order to solve this dilemma?

(Note to instructors: This type of situation is not directly addressed in the text, but the problem Harry is having is that his payment is too small to ever pay off the loan. The interest each month is more than his payment, so the amount he owes is actually growing over time.)

Problem 24

On a coordinate plane sketch the graph of a parabola, given by the equation $y = ax^2 + bx + c$. You may sketch any parabola, but you must give the equation of the parabola you choose and label the vertex and all x or y-intercepts. Be sure that these points are well-labeled on your final sketch.

Problem 25

Three sequences of data are shown below. One sequence can be described by arithmetic growth, one by quadratic growth, and one by geometric growth. Identify the type of growth represented by each sequence, and give a one-sentence reason for your answer in each case.

Sequence 1	Sequence 2	Sequence 3
2.4	3.7	40.
2.51	5.95	32.
2.74	8.2	25.6
3.09	10.45	20.48
3.56	12.7	16.38
4.15	14.95	13.11
4.86	17.2	10.49

Problem 26

On this page, you should write your own problem in which a mixed model can be used to solve a finance problem in which a person either invests money and makes repeated deposits into an account or borrows money and makes repeated payments to reduce the loan amount. You should write the problem clearly, including a statement of the question asked in the problem, and include a solution. Your solution should include both a difference equation and a functional equation, in which you define the relevant variables. For example, if you use the variable b_n in the problem, you must describe what that variable stands for before you use it.

Problem 27

For this problem, assume you are given a very large sheet of paper, as big as three football fields, and assume that the paper is 1/8 inch thick. Let h_n represent the height of the sheet of paper, in inches, when it has been folded in half n times. So $h_0 = 1/8$. When the paper is folded in half once, the height will be $h_1 = 2(1/8) = 1/4$, and when it is folded twice, the height will be $h_2 = 2(1/4) = 1/2$.

a. Write a difference equation and functional equation for h_n:

b. How high will the stack of paper be after 10 folds? (Show work to support your answer)

c. How many times must the paper be folded before the stack is more than 1 mile (= 5280 feet) high?

Problem 28

Students in a particular college dormitory have a choice of two phone plans. On Plan A, students are charged \$11.60 per month for basic service, plus \$0.25 for each call. With Plan B, the monthly charge is \$15.00/month plus \$0.20 per call . Let a_n represent the monthly cost for a student who makes n calls each month under Plan A, and let b_n represent the monthly charge for a student who makes n calls each month under Plan B.

a. Write a difference and functional equation for a_n and b_n:

b. Sketch a graph of the models a_n and b_n on one coordinate plane. Be sure each model is labeled.

c. Describe a guide for students to use to decide which phone plan to choose, depending on the number of calls they plan to make each month. You should be as specific as possible.

Problem 29

Sketch the graph of the quadratic equation $y = -2x^2 + 4x + 4$ on a coordinate plane. Label the coordinates of the vertex, the y-intercept, and any x-intercepts.

Problem 30

A young couple is starting a savings account. They initially put \$2000 in the account, and they plan to deposit \$100 every month. Their bank pays a 5.5% APR (annual percentage rate), compounded monthly. Let b_n represent the balance in the account n months after they open it.

a. Write a functional equation and difference equation for b_n.

b. How much will they have in the account after 10 years? Round your answer to the nearest cent.

c. How much should they deposit each month if they want to have \$30,000 in ten years (assuming their initial deposit is still \$2000)?

Problem 31

For this problem, you should write your own word problem, which can be solved using an *arithmetic* growth model. You should state the problem clearly, including the question to be answered, and then include a well-written solution. If you use a variable, such as a_n in your solution, you should define what the variable stands for before you use it.

Problem 32

The city of Rampa currently has 45,000 people and is growing at a rate of 5,000 people per year. The neighboring city of Brewster currently has 58,000 people and is growing by 3,000 people per year. Let r_n and b_n represent the populations of Rampa and Brewster, respectively, n years into the future. Develop a difference equation and a functional equation to model the growth of both cities. Assuming this type of growth continues, what will the population of each city be in 5 years? When will Rampa have a larger population than Brewster? Be sure to explain your answers fully. On a coordinate plane, sketch a continuous graph of the models for both cities.

Problem 33

Sketch the graph of one of the following quadratic equations. Circle the equation that you choose to graph. On the graph, clearly label both coordinates of the vertex, as well as any x or y-intercepts.

$$\begin{array}{lrcr} \text{Choice 1:} & y & = & -4x^2+3x+2 \\ \text{Choice 2:} & y & = & x^2-4x+9 \\ \text{Choice 3:} & y & = & x^2+4x+4 \end{array}$$

Problem 34

On this page, write a short essay (1-2 paragraphs) in which you discuss logistic growth. Include an explanation of situations which logistic growth is used to model, why it is used, and how the difference equation is derived. (You may choose to discuss aspects of chaos theory as it relates to logistic growth, but this is not required.)

Problem 35

You are working with a friend on a math problem, and to solve the problem, you need to know what $\ln 2$ is. Neither of you have a calculator, but your friend says "at least we know $\ln 2$ is between 0 and

1 - that's a close enough estimate." Assuming your friend did not get this answer from a calculator, explain how she knows this.

Problem 36

In a new housing development, houses are selling for \$100,000. A buyer makes a \$10,000 down payment and is going to take out a \$90,000 mortgage. He has a choice of a 15 year mortgage which charges a 9% annual rate compounded monthly or a 30 year mortgage which charges a 10.75% annual rate compounded monthly. For each of the two options, determine the monthly payment to pay off the loan.

Problem 37

Consider the following excerpt from your textbook:

"So let us consider that typical example again: a population which increases by 10 percent each year. For concreteness, we will imagine that this is a population of fish in a lake. At some point, perhaps there are 10,000 fish in the lake. Then, using the ideas developed in Chapter 9, we can compute the population after n years as $p_n = 10,000 \cdot 1.1^n$. After 5 years the population would be $10,000 \cdot 1.1^5 = 16,105$, after 10 years $10,000 \cdot 1.1^{10} = 25,937$, and so on. We can use the model to predict as far into the future as we please, and the further we look ahead, the larger the population will grow. There is no limit to the size of the population in this model. This is unrealistic in the long run, because it ignores forces that act to limit the size of a population — forces such as competition for food and living space. These forces may be safely ignored with little error when the population is small. But as the population grows larger, as competition for resources becomes more pronounced, it becomes unrealistic to expect population to grow by the same percentage that occurred when the population was small. "

This paragraph describes why a geometric growth model, which has a difference equation of the form $a_{n+1} = ga_n$ for a constant growth factor g, does not realistically describe the behavior of some physical systems such as population growth. Write a one or two paragraph description of the methods we used in class to develop models for these situations, including a discussion of the difference equation we used and how we derived it. What kind of growth is this called?

Problem 38

In a fast food restaurant, milkshakes are made by mixing ice milk with flavored syrup in a refrigerated container, and shakes are drawn through a hose from this container. Assume that the container holds 10 gallons, or 160 cups, of liquid. Accidentally, an employee mixes 1 cup of strawberry syrup into a container already filled with chocolate shake mix, which produces a choco-strawberry shake with a bad taste. The only way to correct the problem is to empty the container, throw away the mix, and refill the container with chocolate shake. Unfortunately, because of the sticky nature of shakes, when the container is emptied, one-tenth of the mixture stays on the sides and is mixed with the added chocolate shake. So, for example, when the container is first emptied, one-tenth of it (16 cups) remains, leaving one-tenth of the strawberry syrup behind also. Let s_n represent the number of cups of strawberry syrup left in the tank after it has been emptied and refilled n times. Write a difference equation and functional equation for s_n. If the tank is considered 'clean' when there is less than 0.125 cups of strawberry syrup left in the tank, how many times must it be emptied and refilled?

Problem 39

In a certain state lottery, winners have a choice of being paid the entire payoff at once or receiving equal monthly installments over 20 years. So if the jackpot is $5,000,000, a winner can take the entire amount initially or choose to receive checks for $20,833.33 each month for twenty years. Assume a particular winner invests her money in a bank which pays a 7.3% APR (annual percentage rate), compounded monthly.

a. If this winner chooses to take the entire jackpot at once and invest it in her bank, how much interest will she earn the first month?

b. If she does not spend any of the money, but lets it grow, how much will she have in the account after 20 years (still assuming she accepted the entire jackpot at once)? You may want to write a difference equation and functional equation for b_n, the amount of money in the account after n months, but this is not required.

c. If she chooses the second option, she will have $20,833.33 deposited in her account initially and every month for 240 months. If she does not withdraw any of this money, how much will she have after 20 years?

Problem 40

A backpack manufacturer is considering selling his backpacks for a price between $20 and $30. His economic consultants provide him with the following equation relating profit to the price, p:

$$\text{Profit} = -200p^2 + 11400p - 98000$$

Based on this equation, what price would you suggest to the manufacturer to yield the maximum profit? How much profit will he make if he charges the price you suggest? (You should be very specific here)

Problem 41

Three quadratic equations are given below. The graph of each equation is a parabola. One of the parabolas has exactly two x–intercepts, one has exactly one x–intercept, and one has no x–intercepts. Identify the number of x–intercepts for each equation below, and give a brief reason for your answer. Note: it is not necessary for you to find the value of the intercepts. You may use a graphing calculator to decide your answers, but your written reason should explain how the equation itself can be used to determine the number of x–intercepts.

$$\begin{array}{lrcl} \text{Equation 1:} & y & = & 3x^2 + 6x + 12 \\ \text{Equation 2:} & y & = & -3x^2 + 6x + 12 \\ \text{Equation 3:} & y & = & -x^2 + 6x - 9 \end{array}$$

3 Sample Reading Assignments by Angela Hare

The following are examples of Reading Assignments which might be used in addition to the *Reading Comprehension* questions provided at the beginning of the exercises for each chapter.

Chapters 1 and 2

Write a short essay, less than two pages, in which you summarize the material in chapters 1 and 2. Specifically, you should mention the *numerical approach,* the *graphical approach,* and the *theoretical approach* and how they are used to analyze a collection of data. What are some advantages and disadvantages of each approach? How are *difference equations* and *functional equations* used? Think of a collection of data, different from the examples in the text, that you might find in a field you are interested in (perhaps your major) and discuss the types of questions you could answer by analyzing this data using the methods of chapter 1.

Chapter 3

Read chapter 3 several times, and as you read, think about how you would teach the material in this chapter to someone else. Write a brief outline of your *lesson plan* for the chapter, as if you were going to teach the chapter to a class of students. Your outline should include key ideas and terms, definitions, and notes about activities you would plan to help the student or class learn the material. You may want to consider some of the things I do in class, but do not simply outline how I presented chapter 3. Be creative!

Chapter 6

Write a one paragraph answer to each of the questions below:

1. What are three differences between arithmetic growth and quadratic growth?
2. Using the coefficients a,b, and c in the equation $y = ax^2 + bx + c$, you should be able to find several specific points on the graph which represents this equation. Name at least three of these points, and describe how you can find them using the coefficients.

Chapter 9

In section 9.1, the author discusses the growth of a population of mice. Explain, in your own words, why arithmetic growth does not describe the growth of this population but geometric growth does describe it. Also, describe two other examples of situations which are modeled well by a geometric growth model. In each case, specifically mention why a geometric model is appropriate.

Chapter 10

1) Chapter 10 has the following major subheadings: Graphs, Algebraic Properties, Solving Equations, and The Number e. Write a 2–3 sentence summary of each of these sections, in which you describe the main ideas mentioned, without taking sentences straight from the text.

2) You and a friend are working on a math project in which you need to find $\log 150$. You don't have a calculator, and your friend says, "At least we know $\log 150$ is between 2 and 3, closer to 2." Explain in a sentence how your friend knows this without access to a calculator.

Chapter 13

Your mother has just written you a letter, in which she describes a fish pond that was installed in your backyard six months ago. She stocked the pond with 6 goldfish (3 pairs) when it was first opened, and she hoped to have 100 fish eventually. Here's an excerpt from her letter:

"The first 3 pairs reproduced quickly. After the first month, they each had 4 babies, which meant the number of fish tripled in the first month. I expected that pattern to continue, so the pond would exceed 100 fish in just three months. But after 2 months, the 18 fish from the first month had only increased to 27, instead of tripling. I wonder why? Your father says some of the adults may be eating the new babies (do you wish I did that to your brother?!). I'm afraid to feed them more, because I've heard that extra food can pollute the water. At this rate, maybe I'll never have 100 fish. Any ideas? Do you ever study problems like this in your Elementary Math Models Class?"

Your assignment is to write a letter back to your Mom, explaining what you think is happening in the pond, and when you predict it will reach 100 fish, if ever. What do you suggest she do? Remember, your Mom doesn't like really short letters! Write your letter on the back of this sheet.