

Entry Level College Mathematics: Algebra or Modeling

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Abstract

In the past several years there has been increasing discussion of elementary mathematical modeling as an entry level college course. In several institutions, modeling is now offered as an alternative to the more traditional college algebra course, and students can choose to complete a modeling course in fulfillment of a general education requirement. Of course, not everyone agrees with this approach. How can math teachers make an informed choice between college algebra and modeling? This paper argues that no such choice is necessary, for many of the instructional goals of the college algebra course can be addressed in a modeling course.

College Algebra or Mathematical Models?

The entry-level college mathematics course in many institutions has for many years been college algebra. Typically, this is the lowest level course that is taught for college credit, and with which students can fulfill a general education requirement in mathematics. In the past decade, however, an increasing number of schools have introduced courses in mathematical modeling at this same level.

In another article in this issue of The AMATYC Review, Lumley (2001), describes the evolution of these modeling courses. As discussed in that article,

there have been ongoing discussions regarding the merits of college algebra versus modeling courses. For non-science and engineering students, modeling courses are supposed to be more interesting to students and cover material that is more applicable in their other courses. On the other hand, some mathematics teachers have been concerned that replacing college algebra with a modeling course might water down the curriculum.

The thesis of this paper is that there is no need to choose between the benefits of a modeling course and the mathematical skills of the traditional college algebra course -- it is possible to have both. A modeling course can easily be designed with many of the same instructional goals as a traditional college algebra course. But the modeling approach provides a context in which the algebraic methods become more meaningful and applicable.

To illustrate these ideas, I will present below a progression of three topics from a modeling course I developed at American University in the mid 1990's. The course is called Elementary Mathematical Models (EMM). For a detailed look at the content and organization, refer to the companion text (Kalman, 1997). The main emphasis of the presentation here will be on mathematical topics that are common to both the modeling course and the traditional college algebra course.

It should be emphasized that there are many variations on entry level mathematics curricula. In some institutions college algebra is the most common choice to fulfill a general education requirement. In others, college algebra is not permitted to count for general education credit. Moreover, there is considerable variety in the structure and content of college algebra courses, and mathematical modeling in one form or another appears in an increasing number of college algebra texts. Variety exists, as well, among entry level modeling courses. It is not my purpose here to survey all of these possibilities, nor to review their relative merits. Rather, the goal is to demonstrate in one specific example how core elements of a traditional college algebra curriculum arise in a course with a narrow focus on modeling. In this way, I hope to encourage college algebra teachers to consider a modeling alternative, and to seek out the core algebra topics that are generally to be found in modeling curricula.

Before proceeding to the main discussion, a short account of the philosophy and rationale for EMM will be presented.

Rationale and Philosophy of EMM

The EMM course seeks to serve the same students as traditional college algebra courses. These are generally students who have completed three years of college preparatory mathematics in high school, but who are not headed for calculus. They may need to take a mathematics course for a general education requirement. They may also need familiarity with the main ideas of college algebra for quantitative aspects of general education courses in areas outside of mathematics, such as science, economics, and business. For most of these students this will be the only mathematics course completed in college.

The development of EMM was guided by two main goals. The first was to create a course that students would perceive as intrinsically interesting and worthwhile. The EMM course tries to present a coherent story throughout the semester, and to convey something of the utility, power, and method of mathematics. Among other things, this depends on genuine and understandable applications.

The second main goal was to emphasize the topics that students are most likely to meet in mathematical applications in other disciplines. I am thinking here of the most elementary applications, formulated in terms of arithmetic and simple algebraic operations: linear, quadratic, polynomial, and rational functions; square roots; exponential and logarithmic functions. These functions are the building blocks for the simple models that appear in first courses in the physical, life, and social sciences. They are also the central focus for the traditional college algebra course. That is why EMM shares so many of the instructional goals of college algebra.

Combining the two goals is complicated by the diversity of the student audience. Entry level college math courses serve students with a broad range of backgrounds. At one extreme are students for whom this course is a first exposure to college level mathematics. At the other extreme are students who have studied more advanced topics (even calculus) and are looking to refresh rusty skills, or for an easy way to fulfill a requirement. The challenge is to make the material fresh and interesting for the more advanced students, and at the same time, accessible as a first exposure. EMM attempts to do this by introducing each mathematical topic in the context of an application. The applications are all analyzed using a common methodology, involving difference equations. The difference equation methods are applicable to a number of problems that have obvious significance and relevance. The desired mathematical topics come up in a natural way as difference equations are used to study the applications.

The algebraic emphasis in EMM is restricted to what is really required for working with these simple models. Here there is a true departure from the traditional college algebra course. For example, logarithms are used to solve simple exponential equations, but only natural and base 10 logarithms are discussed. Equations involving several logarithmic terms with arbitrary or unknown bases, a standard exercise in college algebra, are not covered in EMM. Similarly, square roots and fractional exponents are included in EMM, but not radical expressions for indices other than 2.

As these examples suggest, most of the traditional topics of college algebra appear in EMM, but they are not carried as far. This is a deliberate decision based on the two goals mentioned earlier. By focusing only on the mathematical manipulations that appear in realistic and understandable applications, EMM can convincingly demonstrate the utility of the algebra that does appear. At the same time, there is an increased emphasis on conceptual understanding of how mathematics really does get applied, and fundamental aspects of the modeling framework.

Sample Progression of Three Topics

As mentioned at the outset, the main point of this paper is to show that a modeling course can include a great deal of standard college algebra. Toward that end, a progression of three topics from EMM will be presented, with a discussion of the algebra that goes along with these topics. The discussion is not intended to be exhaustive -- there are many other topics in EMM, and many other related areas of college algebra, beyond what is considered here. This presentation should be considered merely as a sample of college algebra material that arises naturally in a modeling course.

The three topics discussed below are *Geometric Growth*, *Mixed Geometric and Arithmetic Growth*, and *Logistic Growth*. To provide an overall context for these topics, there follows a brief discussion of difference equations, and the methods of analysis that are emphasized in EMM, as well as some general remarks relating this discussion to college algebra. Then the narrative will proceed to the three sample topics.

Difference Equations

Difference equations concern recursive patterns in number sequences. For example, in the famous Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, ... , each term is the

sum of the two preceding terms. That is a recursive pattern because it can be applied repeatedly, always using terms just computed to find the succeeding terms.

Recursive patterns can be represented symbolically using difference equations. Let f_n be the n^{th} term of the Fibonacci sequence. Since each term is the sum of the two preceding terms, we have

$$f_{n+1} = f_n + f_{n-1}.$$

That is a difference equation -- it expresses a term of the sequence as a function of preceding terms (and, in some examples, of the index n .)

As another example, consider the sequence 1, 4, 9, 16, 25, Most math teachers immediately recognize this as the sequence of perfect squares, so that the n^{th} term is n^2 . But at the level of college algebra, students are more likely to spot a recursive pattern: the differences between adjacent terms are 3, 5, 7, 9, ..., successive odd integers. According to this recursive pattern, the next term after 25 is found by adding the next successive odd integer, 11, to obtain 36. If we denote by a_n the n^{th} term of this sequence, then the difference equation can be expressed as

$$a_{n+1} = a_n + 2n + 1.$$

In contrast, the perfect squares interpretation is represented

$$a_n = n^2.$$

This expresses a_n as a function of the index n and gives the *solution* of the difference equation. Typically, although these solutions are derived with n restricted to integer values, the problem context makes the solutions meaningful for fractional values of n as well. In particular, for solving equations and graphing, n should be thought of as a real variable.

Recursive patterns and their analysis are a common theme running through EMM. The main idea is that recursive patterns, expressed as difference equations, are natural to observe or hypothesize, while equations for the solutions of difference equations are convenient for analysis. The formulation and solution of difference equations thus provides a general method that can be used in many applications.

The simplest kinds of recursive pattern lead directly to the elementary functions studied in college algebra. As a first example we have arithmetic

growth, characterized by a recursive pattern in which each term is found by adding a fixed amount to the preceding term. This produces an arithmetic progression. The solution to the difference equation is a linear function of n . A slightly more complicated pattern is one in which the differences between successive terms is not constant, but follows an arithmetic growth pattern. That is what we saw in the perfect square example, where the differences are successive odd integers. The solution to this kind of difference equation is a quadratic function of n . Generalizing arithmetic growth in a different way, geometric growth occurs when successive terms increase or decrease by a constant percentage, or equivalently, through multiplication by a fixed constant. This gives rise to a geometric progression, and the solution is an exponential function. Geometric and arithmetic growth can be combined in a general linear model, with each term a linear function of the preceding term. This is what was referred to earlier as *mixed arithmetic and geometric growth*. The solution is an exponential function plus a constant. Another kind of model, *logistic growth*, is a generalized form of geometric growth in which successive terms increase by a percentage that is not constant, but depends linearly on the preceding term. This kind of recursive pattern arises naturally in population models.

Analysis of Difference Equations

Some common methods of analysis apply in all of the difference equation models described above. One approach is direct numerical experimentation and exploration. Algebra is used in this context to express general relationships. Part of the experimentation also involves identifying properties of families of functions, *e.g.* linear functions for arithmetic growth models, or exponential functions for geometric growth models. Learning about properties of families of functions is thus a goal of instruction that is common to both EMM and college algebra.

Another recurring analysis method is fitting a model to actual data by choosing the best values for certain parameters. Just formulating this problem symbolically, and in particular, understanding how and why parameters are used in the model, stresses important concepts from college algebra.

Direct prediction is a third common method of analysis for difference equation models. This is simply computing a_n for specified values of n . In EMM this is approached numerically, graphically, and symbolically. In particular, when a solution is known for the difference equation, so that $a_n = f(n)$, direct prediction involves function evaluation and reinforces important aspects of the function concept and notation.

As a final common analysis method, many problems require *inverse prediction*. Here the goal is to predict when a particular value of a_n will be observed. Again thinking of the solution $a_n = f(n)$, inverse prediction amounts to specifying the value of f and trying to determine n . That is equivalent to inverting the function f . In EMM, inverse prediction is also addressed numerically, graphically, and symbolically. All of these involve skills that are stressed in college algebra. In particular, the symbolic approach concerns solving equations, a central focus for college algebra.

The foregoing material has provided a general description of difference equations and corresponding methods of analysis. From the general description some of the connections between difference equation models and college algebra topics should already be apparent. We proceed now to outline three specific difference equation topics, highlighting the college algebra ideas that arise along the way. Note that these topics are not from the beginning of the EMM course. Students already have a good deal of experience with difference equation methods before they reach the topics below.

Geometric Growth

As already indicated, geometric growth refers to a recursive pattern in which each term is a fixed multiple of the preceding term. This kind of model has many applications, including population growth, compound interest, radioactive decay, drug elimination/metabolization, and passive cooling/heating. A typical difference equation is given by

$$P_{n+1} = 1.2 P_n$$

for a population that grows by 20% from one term to the next. The solution to this difference equation is

$$P_n = P_0 \cdot 1.2^n$$

where P_0 is the initial population size. A sample direct prediction question would be *What will the population be in year 8?* A sample inverse prediction question is *When will the population reach 10,000?*

What topics and skills from college algebra are covered in a unit on geometric growth? Properties of exponential functions Ab^n are heavily emphasized. That involves graphs, the significance of the parameters A and b , solving equations, and the meaning and use of logarithms. All of these are standard topics in college algebra, as well.

Mixed Arithmetic and Geometric Growth

In mixed growth models, each term combines a fixed multiple of the preceding term with a fixed increment. This kind of pattern arises naturally in several settings, including amortized loans, installment savings, repeated drug doses, chemical reactions, and pollution. In general, mixed growth is characterized by a difference equation of the form

$$a_{n+1} = r a_n + d$$

where r and d are numerical constants. The solution to this difference equation has the form

$$a_n = A \cdot r^n + C$$

where the constants A and C depend on the initial value a_0 and the parameters r and d . When $r < 1$ there is a horizontal asymptote at C . This represents an equilibrium value in the model.

As an example, consider a model for the pollution in a lake. Assume that pollution flows into the lake from man-made sources, while clean water flows into the lake at one end, and the polluted water flows out of the lake at the other. Under these conditions, it is reasonable to expect that the outflow of pollution occurs in proportion to the existing concentration in the lake, while the inflow of pollution occurs at a constant rate. To be specific, suppose that in each unit of time, one tenth of the pollution already in the lake flows out, and that three more units of pollution are added. The difference equation describing this situation is

$$p_{n+1} = .9 p_n + 3$$

and the solution is

$$p_n = (p_0 - 30) (.9^n) + 30.$$

Typical questions that might be asked are *What will the pollution level be in year 8? When will the pollution level reach 100? What will happen in the long term?*

A unit on mixed models provides many opportunities to introduce and reinforce ideas from college algebra. As a first step toward illustrating this, let us

consider the derivation of a solution. In EMM this is first introduced in terms of patterns. In fact, by this point in the EMM course, students have seen inductive patterns used many times to discover the solution to a difference equation. Here is what the pattern looks like for the pollution example. First, assuming a starting value of 20, several terms of the sequence are worked out from the difference equation:

$$p_0 = 20$$

$$p_1 = 20(.9) + 3$$

$$p_2 = 20(.9^2) + 3(.9+1)$$

$$p_3 = 20(.9^3) + 3(.9^2+.9+1)$$

Here, the right side of each new equation is obtained by carrying out the recursive pattern: multiply the previous entry by .9 and add 3. But the operations are not carried out numerically. Rather, the numbers are manipulated symbolically. Now the pattern that emerges is pretty clear, and most students will be able to observe and use that pattern. So, for example, if you ask them to predict the entry for $n = 8$, they will respond correctly with

$$p_8 = 20(.9^8) + 3(.9^7+.9^6+ \dots+.9+1)$$

Based on that pattern, the solution is given as

$$p_n = 20(.9^n) + 3(.9^{n-1}+.9^{n-2}+ \dots+.9+1)$$

This expression is further simplified using the formula for summing a geometric progression. That leads to

$$p_n = 20(.9^n) + 3\left(\frac{1-.9^n}{1-.9}\right).$$

This is a model for the general solution to a mixed growth model difference equation, given by

$$p_n = p_0(r^n) + d\left(\frac{1-r^n}{1-r}\right).$$

This is a natural form for the equation, given the derivation, but it is not the most convenient form to work with. Using the lake example again, the solution can be rearranged to the equivalent equation

$$p_n = 30 - 10(.9^n).$$

Of course, algebraically rearranging one form to arrive at a different form is one of the most fundamental uses of algebra. Here it occurs in a way that makes the utility self-evident. The derivation of the solution leads us most naturally to express it in one form, but to use it we prefer a different form. This kind of context for symbolic manipulation is unfortunately absent in many college algebra curricula. Students are drilled on abstract problems in which one form is given and another is desired, but there is no indication how such a problem arises. In any case, the EMM course is full of situations where this kind of algebraic rearrangement arises in a natural way, and so provides many opportunities for students to master the techniques of symbolic manipulation, while observing why those techniques are needed.

As discussed previously, one of the standard questions that arises in EMM calls for inverting the solution to a difference equation. For the example before us, a typical question would be *When will the pollution level reach 100?* That becomes the equation

$$30 - 10(.9^n) = 100.$$

As with earlier topics, students are again encouraged to consider questions like this numerically, graphically, and symbolically. In particular, the symbolic solution requires algebraic rearrangement and the use of logarithms. This topic is certainly a familiar one in college algebra.

Another interesting aspect of the mixed growth model concerns the asymptote or equilibrium value. In the traditional college algebra course, asymptotes are discussed, and in some treatments the limit notation is even used. That approach expresses the equilibrium value for the general mixed model in the form

$$\lim_{n \rightarrow \infty} f(n) = C.$$

The recursive point of view provides an alternative approach to this idea. Imagine that the model actually reaches equilibrium. That means when you apply the recursive procedure (multiply by .9 and add 30) you end up with exactly the same value you already had. Therefore the equilibrium value is a fixed point of the recursive process. In symbols, if x is the equilibrium value, then

$$.9x + 30 = x.$$

Solving this equation shows that the equilibrium value is 30. This is another opportunity to use algebra, and reinforces the material on linear equations encountered earlier in the EMM course.

To summarize, the unit on mixed growth models includes a large number of topics that are also standard fare in the college algebra curriculum. In the course of formulating and analyzing these models students study

- Properties of shifted exponentials $Ab^n + C$
- Graphs, horizontal asymptotes
- Significance of parameters A , b , and C
- Solving equations, logarithms
- Finding fixed points
- Deriving the general solution to the difference equation
- Transforming expressions

All of these involve algebraic skills and ideas common to the college algebra curriculum.

Logistic Growth

Logistic growth is a modified version of geometric growth. Each term is a multiple of the preceding term, but the multiplier varies linearly with the size of the term. For example, in a population model, assume that the population p goes up in a year by a factor of $.01(200 - p)$. When p is small, the growth is approximately multiplication by 2. On the other hand, when p is near 100, the multiplier is nearly 1, so that very little change in the population occurs.

Let's use this model in a recursive setting. We start with some initial population p_0 . The next year's population, according to the model, will be found by multiplying p_0 by the factor $.01(200 - p_0)$. That gives p_1 . In a similar way, to

compute p_2 we multiply p_1 by $.01(200 - p_1)$. This process can be repeated any number of times. The difference equation for the model is $p_{n+1} = .01(200 - p_n)p_n$.

This kind of model reflects the fact that populations cannot grow geometrically for very long before something acts to limit growth. Indeed, the derivation of the model is fairly natural, as soon as it is observed that a constant multiplier in geometric growth is not very realistic. In most situations, we would expect that the multiplier would have to change as the population size increases, thus leading to the idea of a multiplier given by a function of the population size. Adopting the simplest possible function, namely a linear function, produces a logistic growth model.

Logistic growth models as developed here cannot be solved symbolically. That is, there is no elementary expression for p_n as a function of n . (This is in contrast with logistic differential equation models, whose solutions are combinations of rational and exponential functions). Nevertheless, the logistic growth difference equation models have interesting properties that can be discovered graphically and numerically, and then formulated and verified symbolically. In the process, important college algebra topics again make an appearance.

For the general logistic growth difference equation $a_{n+1} = m(L - a_n)a_n$, with positive parameters m and L , negative values of a_n will occur if the population ever exceeds L . Assuming the model is only meaningful for positive a_n , L is an absolute limit on the size of the population. The product mL determines the long term behavior of the model. As long as mL is between 0 and 4, if the initial value is between 0 and L , then all succeeding values similarly remain between 0 and L . For $0 < mL < 1$, the population always dies out, approaching a value of 0 asymptotically. For $1 \leq mL < 3$, the population approaches an equilibrium value of $L - 1/m$. This can be derived as a fixed point of the recursion, just as in the case of mixed growth models. For $3 \leq mL < 3.61547\dots$, the population will approach an oscillation among two or more fixed values. Finally, for $3.61547\dots \leq mL < 4$, the future behavior of the model may be chaotic. Indeed, the logistic model is one of the simplest examples considered in courses on chaos, and the emergence of chaotic behavior as mL is increased is a virtual paradigm for the onset of chaos in discrete systems.

Some of these results are not really accessible in a course at the level of college algebra. For example, the existence and identity of the number 3.61547... requires mathematical methods of calculus (Gulick, 1992, pp 41-50). But some of the results can be derived using basic algebra, including properties of quadratic

functions. It will be too great a digression to go into those topics in detail here, but in general terms the analysis involves intersections of parabolic and linear graphs, and some simple inequalities.

A variant on logistic growth, valid for $mL < 3$, involves the introduction of harvesting. Imagine that the example we considered before is a logistic growth model for fish in a lake, with p_n given in units of thousands. There is an equilibrium value of 100,000 fish, because the model levels off to a fixed point of 100. The model predicts that the fish population will approach this equilibrium value and then remain stable at that level. Suppose it is decided to harvest some of these fish, say 20,000, each recursion cycle. We can add that effect to the model by simply using the earlier recursion, and then subtracting 20 at the end. That leads to the difference equation

$$p_{n+1} = .01(200 - p_n)p_n - 20.$$

More generally, the difference equation for logistic growth with modeling always takes the form $a_{n+1} = m(L - a_n)a_n - h$.

Introducing harvesting in this way modifies the long term predictions for the model. There is an interesting interaction between the amount harvested and the initial value for the model. The fixed points of the recursion are a key to the analysis. The fixed point equation is

$$m(L - x)x - h = x,$$

or in simplified form,

$$mx^2 + (1 - mL)x + h = 0.$$

Solving this equation to find the fixed points provides a nice review and application of quadratic equations. Suppose the roots are r and s , with $r < s$. If the initial population is less than r , or greater than $L - r$, then the model predicts that the population will die out. Otherwise, for an initial population between r and $L - r$, the population will approach an equilibrium value of s . Of course, r and s depend on the harvesting level, h . It is not difficult to see that there is a critical value of h that limits the amount that can be harvested. Harvesting more than the critical amount will inevitably kill off the population. Below the critical level, the more you harvest, the smaller the interval $[r, L - r]$ to which the initial population must be confined.

The EMM course does not derive all of these conclusions for the students, but it does present them and show how to interpret them in the context of specific models. More generally, the logistic growth models, with and without harvesting, make extensive use of graphical methods, as well as properties of quadratic functions, solving linear and quadratic equations, and simple inequalities. These are all topics that appear in college algebra.

Conclusion

One of the main objectives of this paper has been to show the similarity between the mathematical topics covered in college algebra and those covered in a modeling course like EMM. As a secondary goal, the motivations for placing those topics into a modeling context have been presented. In general terms, the rationale is to present the core college algebra topics in a context that shows off their value and relevance. In the process, some aspects of the traditional college algebra curriculum may be de-emphasized or eliminated, and some teachers may fear that this will lead to a watering down of the course. That need not be the case. As the material on logistic growth demonstrates, there are sophisticated and substantive quantitative issues that arise naturally in the context of simple difference equation models. In EMM, for example, the reduced demands for abstract symbolic manipulation are more than compensated for by the increased emphasis on quantitative reasoning in realistic contexts.

Overall, EMM is still primarily concerned with the same mathematical topics as college algebra, in spite of differences of approach or emphasis. Indeed, one of the main goals of EMM is to convey to students the idea that algebra is an important and powerful tool. By emphasizing models and applications, EMM attempts to let students reach this conclusion themselves, from seeing algebra in action. The algebra that does appear in the course is presented in a way that makes it clear *why* algebra is needed, and what it contributes to formulating and analyzing models. In this way, EMM exemplifies one viable prescription for entry level college mathematics: algebra **AND** modeling.

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