Euler, Dilog, and the Basel Problem

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The Basel Problem

- Determine the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$
- Well known history: Euler's solution $= \pi^2/6$
- Approximately Avogadro's number of proofs now known
- This talk: standard power series methods lead to a simple proof, using the *dilog* function
- Euler again plays a central role
- Did he know this proof?

Outline

I. Solving the Basel Problem with power series and dilog

II. Dotting i's and crossing t's: rigorous justification for the proof's formal manipulations

III. The historical Question: Did Euler know this proof?

Part I

A power series approach to the Basel Problem

Simple Power Series Methods

- Example: Determine $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$
- Introduce a power series $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$
- We want $f(-1)$; we know $f(0) = 0$
- $f'(x) = \displaystyle\sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}$
- Therefore $f(x) = \displaystyle\int_0^x \frac{1}{1-t}\, dt = -\ln(1-x)$
- Conclusion: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = f(-1) = -\ln 2$

Power Series and the Basel Problem

- Want $\sum_{k=1}^{\infty} \frac{1}{k^2}$ so consider $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$
- Differentiate and multiply by x : $xf'(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$
- Recognize this series as $-\ln(1-x)$
- $f(x) = \int -\frac{\ln(1-x)}{x}\, dx$
- Constant of integration determined by $f(0) = 0$

Definite Integral Formulation

- $f'(x) = -\dfrac{\ln(1-x)}{x}$
- $f(0) = 0$
- $f(x) = \displaystyle\int_0^x -\frac{\ln(1-t)}{t}\,dt$
- $\displaystyle\lim_{t\to 0} -\frac{\ln(1-t)}{t} = 1$
- Integrand has a removable singularity at $t = 0$
- Conclusion: $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \int_0^1 -\frac{\ln(1-t)}{t}\,dt$

Can We Do the Integral?

- Want $\int_0^1 -\frac{\ln(1-t)}{t}\,dt$
- Usual methods of antidifferentiation seem ineffective
- Try Mathematica: www.quickmath.com
- (switch to internet here)
- If only Euler had Mathematica!

Calculus : Integrate Advanced

EXPRESSION

-log(1-x)/x

VARIABLE(S) & LIMITS

x,0,a

[one per line]

OPTIONS

Output Style

Traditional : 20 points   Text

Reset   Integrate

Command Integrate

Expression

$$-\frac{\log(1-x)}{x}$$

Variables & Limits

$x \quad 0 \quad 1$

Result

$$\frac{\pi^2}{6}$$

How Did Mathematica Do That?

- Look at indefinite integral
- (Back to internet)
- (Google search for Li_2 at Mathworld)

Calculus : Integrate

Integrate -log(1-x)/x with respect to x Integrate

Command
Integrate

Expression

$$-\frac{\log(1-x)}{x}$$

Variables & Limits

x

Result

$\mathrm{Li}_2(x)$

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Search **MathWorld**

Note that the notation **Li_2**(x) is unfortunately similar to that for the logarithmic integral ... Dilworth's Lemma (Wolfram **MathWorld**) **...**

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Dilogarithm -- from Wolfram **MathWorld**

The dilogarithm **Li_2**(z) is a special case of the polylogarithm Li_n(z) **....** **MathWorld**--A Wolfram Web Resource. http://**mathworld**.wolfram.com/Dilogarithm.html **...**

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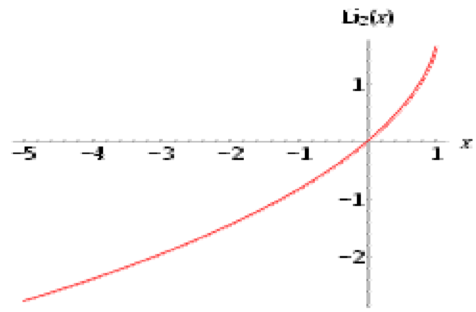
Phorum :: Probabilités :: loi logistique - [Translate this page]

supplément gratuit glané sur **mathworld**: $$\int_0^{+\infty} \frac{tdt}{1+e^t}=-\Gamma (2)**Li_2**(-1)$$ avec $**Li_2**$ fonction polylogarithme d'ordre 2 et **...**

les-mathematiques.u-strasbg.fr/phorum5/read.php?12,365661,365755 - 89k - Cached - Similar pages

Done

Dilogarithm



The dilogarithm $\mathrm{Li}_2(z)$ is a special case of the polylogarithm $\mathrm{Li}_n(z)$ for $n = 2$. Note that the notation $\mathrm{Li}_2(x)$ is unfortunately similar to that for the logarithmic integral $\mathrm{Li}(x)$. There are also two different commonly encountered normalizations for the $\mathrm{Li}_2(z)$ function, both denoted $L(z)$, and one of which is known as the Rogers L-function.

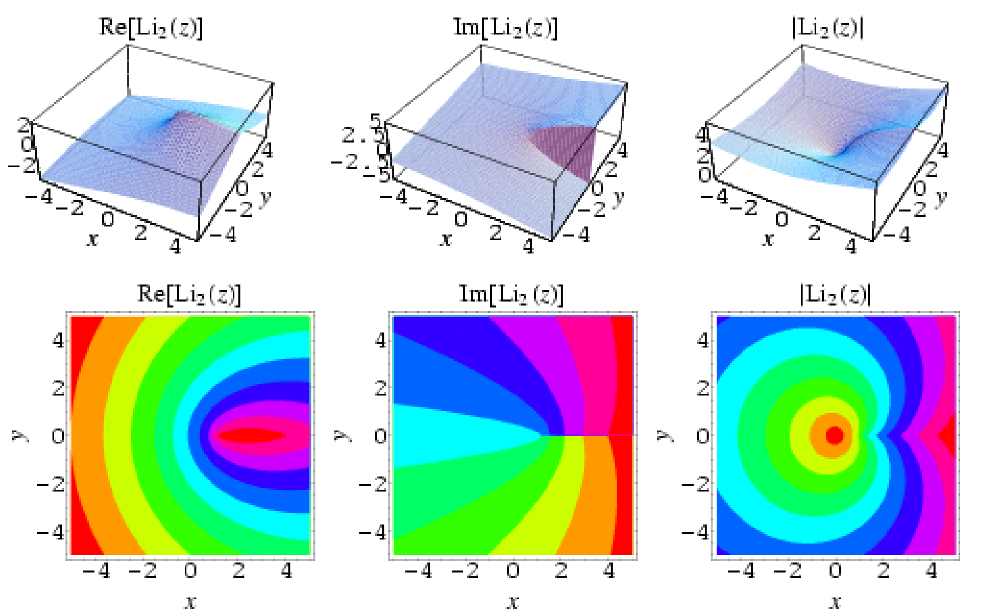
The dilogarithm is implemented in *Mathematica* as `PolyLog[2, z]`.

The dilogarithm can be defined by the sum

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \qquad (1)$$

or the integral

$$\mathrm{Li}_2(z) \equiv \int_z^0 \frac{\ln(1-t)\,dt}{t}. \qquad (2)$$



Plots of $\mathrm{Li}_2(z)$ in the complex plane are illustrated above.

How Did Mathematica Do That?

- Look at indefinite integral
- (Back to internet)
- (Google search for Li_2 at Mathworld)
- Circular Logic: we cannot use this approach to solve the Basel Problem without some other way to evaluate the integral
- Have we reached a dead-end?
- NO!!!!

The Dilog Function

- Define dilog function by $\mathrm{Li}_2(x) = \int_0^x \frac{-\ln(1-t)}{t}\,dt$
- Note $\mathrm{Li}_2'(x) = \frac{-\ln(1-x)}{x}$
- $\mathrm{Li}_2(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$
- $\mathrm{Li}_2(-1/x) + \mathrm{Li}_2(-x) + \frac{1}{2}(\ln x)^2 = C$ (constant)
- Proof: Show the derivative of the left side is 0
- Set $x = 1 : C = 2\mathrm{Li}_2(-1)$
- $C = 2(-1 + 1/4 - 1/9 + 1/16 - \cdots)$

Odd and Even Terms

- Let $S = \sum 1/k^2$
- Even terms: $\sum 1/(2k)^2 = (1/4) \sum 1/k^2 = S/4$
- Odd terms: $S - S/4 = 3S/4$
- $\mathrm{Li}_2(-1) = -1 + 1/4 - 1/9 + 1/16 - \cdots =$ Evens - Odds $= -S/2$
- $\mathrm{Li}_2(-1/x) + \mathrm{Li}_2(-x) + \dfrac{1}{2}(\ln x)^2 = 2\mathrm{Li}_2(-1) = -S$

Play It Again, Sam

- $\mathrm{Li}_2(-1/x) + \mathrm{Li}_2(-x) + \dfrac{1}{2}(\ln x)^2 = 2\mathrm{Li}_2(-1) = -S$
- Set $x = -1$
- $2\mathrm{Li}_2(1) + \dfrac{1}{2}(\ln -1)^2 = -S$
- $2S + \dfrac{1}{2}(\ln -1)^2 = -S$
- $3S = -\dfrac{1}{2}(\ln -1)^2$
- $S = \dfrac{-(\ln -1)^2}{6}$

Finishing the Proof

- $S = \dfrac{-(\ln -1)^2}{6}$
- $-1 = e^{\pi i} \Rightarrow \ln -1 = \pi i$
- $(\ln -1)^2 = -\pi^2 \Rightarrow -(\ln -1)^2 = \pi^2$
- $S = \dfrac{\pi^2}{6}$

Comments

- Two key ideas make the proof work – the dilog identity and $e^{\pi i} = -1$ – and Euler gave us BOTH!

- The argument given is in the Euler style – lots of formal manipulation.

- Complete rigor depends on careful discussion of convergence and of definition of complex integrals and logarithm

- Historical Question: Was Euler aware of this method for solving the Basel problem?

- The above proof is given by Lewin in his book *Polylogarithms and Associated Functions*, North Holland, 1981. He doesn't claim to have invented the proof, but gives no attribution for it.

Part II

Justifying Formal Manipulations

Complex Functions

- Natural logarithm: principle branch of inverse exponential
- For $z = re^{i\theta}$ with $-\pi < \theta < \pi$, $\ln(z) = \ln r + i\theta$
- Analytic in domain $\mathbb{C} \setminus (-\infty, 0]$, ie, the complement of the real interval $(-\infty, 0]$ in the complex plane
- $\ln(1-z)$ has domain $\Omega = \mathbb{C} \setminus [1, \infty)$
- For $z \in \Omega$, define $F(z) = \begin{cases} 1 & \text{if } z = 0 \\ \frac{-\ln(1-z)}{z} & \text{otherwise.} \end{cases}$
- Differentiability of F at the origin established by direct computation of the limit defining $F'(0)$
- Conclusion: F is analytic in Ω

Definition of Dilog

- Define dilog function for complex variable $z \in \Omega$: $\mathrm{Li}_2(z) = \int_0^z F(w)dw = \int_0^z \frac{-\ln(1-w)}{w} dw$
- Integral represents complex (path) integration
- Within Ω definition of $\mathrm{Li}_2(z)$ path independent

Series Expansion at 0 for Dilog

- $-\ln(1-z) = \frac{z}{1} + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots$
- $\mathrm{Li}_2'(z) = F(z) = \frac{1}{1} + \frac{z}{2} + \frac{z^2}{3} + \frac{z^3}{4} + \cdots$
- Integrate term by term: $\mathrm{Li}_2(z) = \frac{z}{1} + \frac{z^2}{2^2} + \frac{z^3}{3^2} + \frac{z^4}{4^2} + \cdots$
- Radius of convergence is 1 for each of these series
- In particular dilog equals its series expansion $|z| < 1$.
- Dilog series converges absolutely for $|z| = 1$, but $z = 1 \notin \Omega$
- Is dilog represented by its series for $|z| = 1; z \neq 1$?

Radial Limits Theorem

- For $g(z) = \sum a_k z^k$
- Radius of convergence 1
- Assume $\sum |a_k|$ converges
- For $|z| = 1$, radial limit means $\lim_{r \to 1^-} g(rz)$
- Theorem: $g(rz) \to \sum a_k z^k$
- Uniformity: $|g(rz) - \sum a_k z^k| < \epsilon$ for $1 - \delta < r < 1$ independent of z

Proof

For $|z| = 1$, $0 < r < 1$, and positive integer N

$$
\begin{aligned}
|\sum a_k z^k - g(rz)| &\leq \sum |a_k z^k (1 - r^k)| \\
&\leq \sum^{N} |a_k|(1 - r^k) + \sum_{k=N+1}^{\infty} |a_k|(1 - r^k) \\
&\leq \sum^{N} M(1 - r^N) + \sum_{k=N+1}^{\infty} |a_k|; \; M = \max_{1 \leq k \leq N} |a_k| \\
&\leq NM(1 - r^N) + \sum_{k=N+1}^{\infty} |a_k|
\end{aligned}
$$

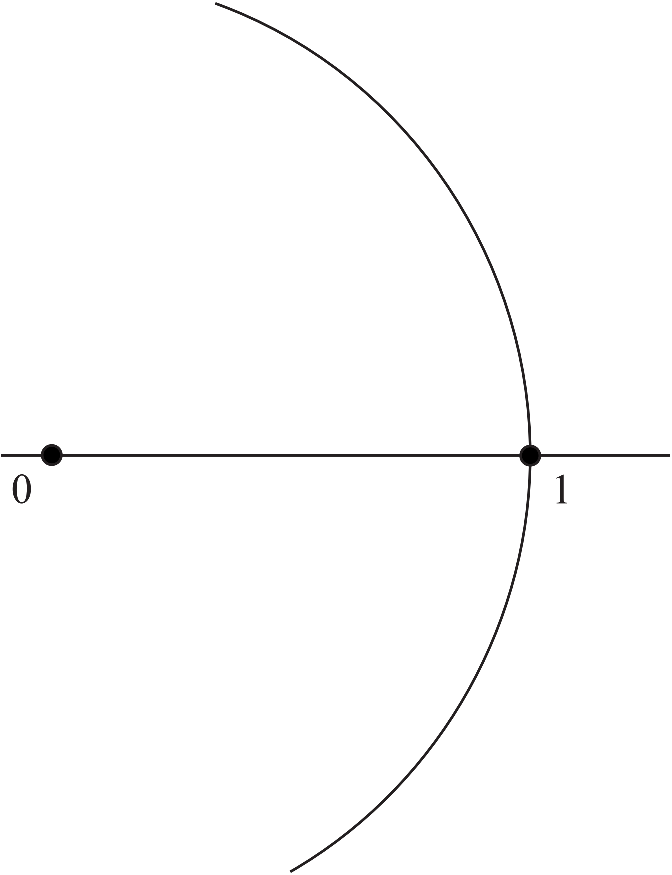
Given $\epsilon > 0$ we can choose N to make the second sum small and then choose r close enough to 1 to make $NM(1 - r^N)$ small.

Application to Dilog

- Apply the preceding result with $a_k = 1/k^2$.
- We know $g(z) = \mathrm{Li}_2(z)$
- Clearly, $\sum z_0^k/k^2$ is absolutely convergent for every z_0 with $|z_0| = 1$.
- By the theorem, $\sum z_0^k/k^2 = \lim_{r\to 1^-} \mathrm{Li}_2(rz_0)$
- Continuity of Li_2 in $\Omega \Rightarrow \sum z_0^k/k^2 = \mathrm{Li}_2(z_0)$ for $z_0 \neq 1$
- Conclusion: We can extend the power series representation of $\mathrm{Li}_2(z)$ to all points of the unit circle, excluding $z = 1$.

Extension to $z = 1$

- Define $\mathrm{Li}_2(1) = S = \sum 1/k^2$
- With this definition, Li_2 is continuous on the closed unit disk U
- We already know Li_2 is analytic (hence continuous) away from $[1, \infty)$, and hence in $U \setminus \{1\}$.
- Continuity at 1 (in U) follows from the radial limits theorem, using the uniformity of δ over z_0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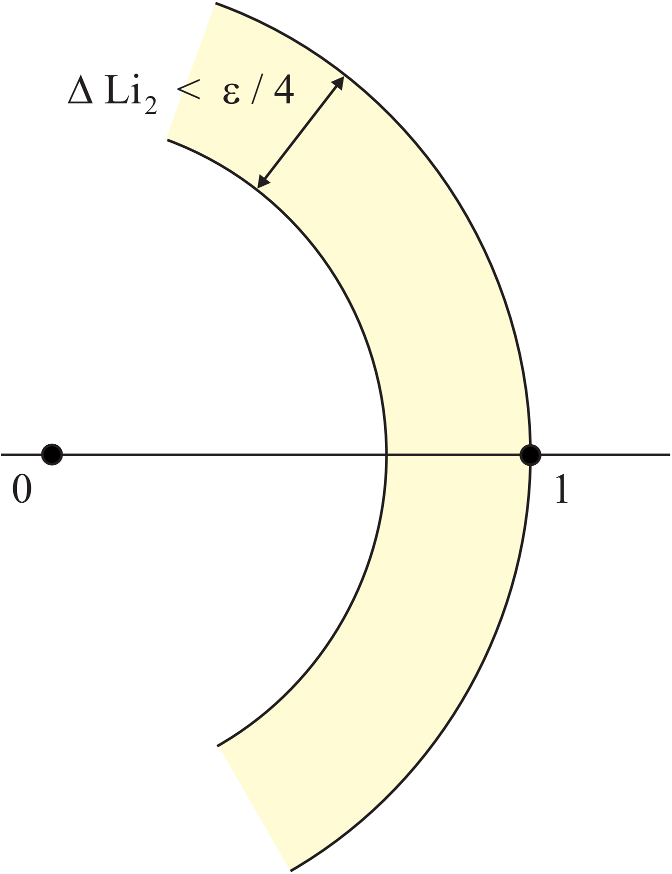

$\Delta \, \mathrm{Li}_2 < \varepsilon / 4$
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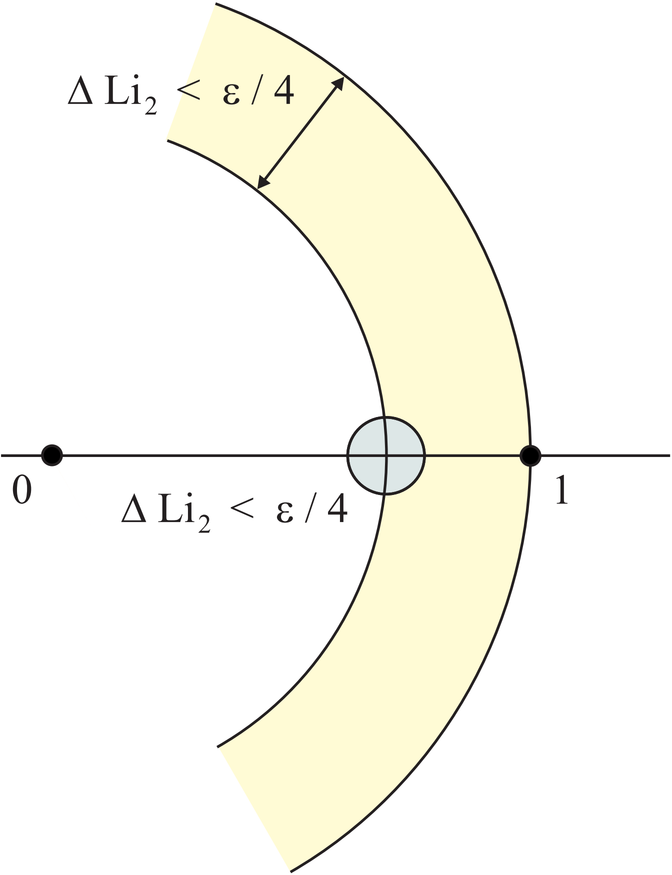
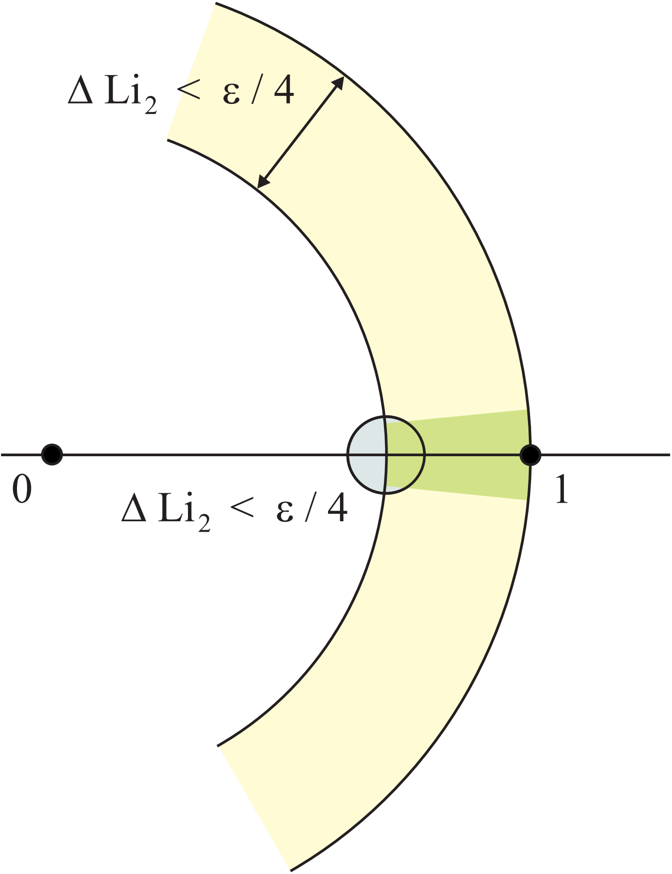
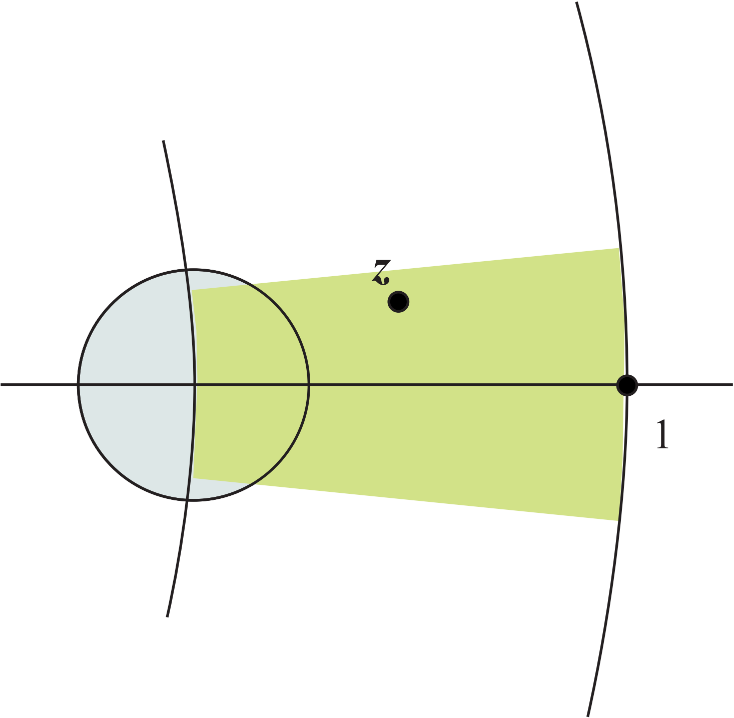


Δ Li2 < ε / 4
0
Δ Li2 < ε / 4
1



Δ Li$_2$ < ε / 4
0
Δ Li$_2$ < ε / 4
1



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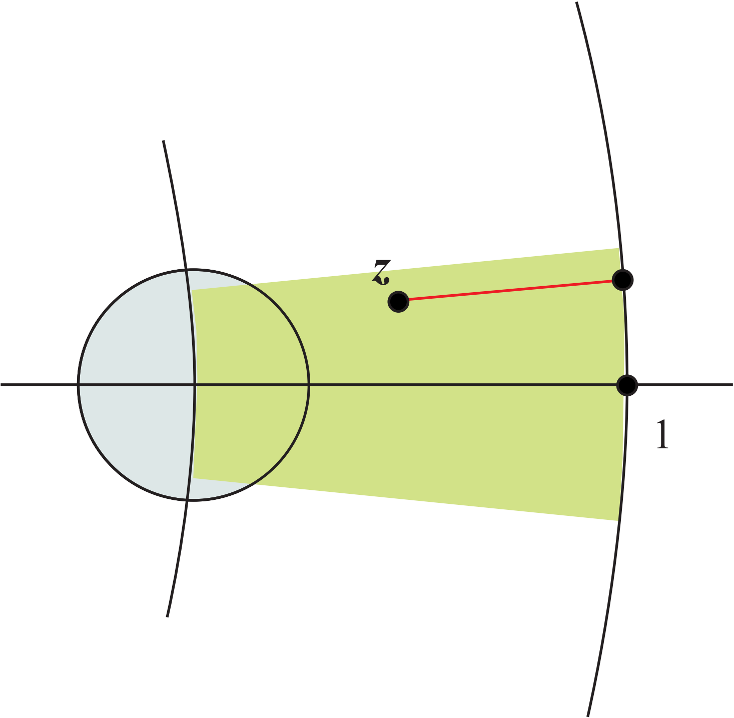
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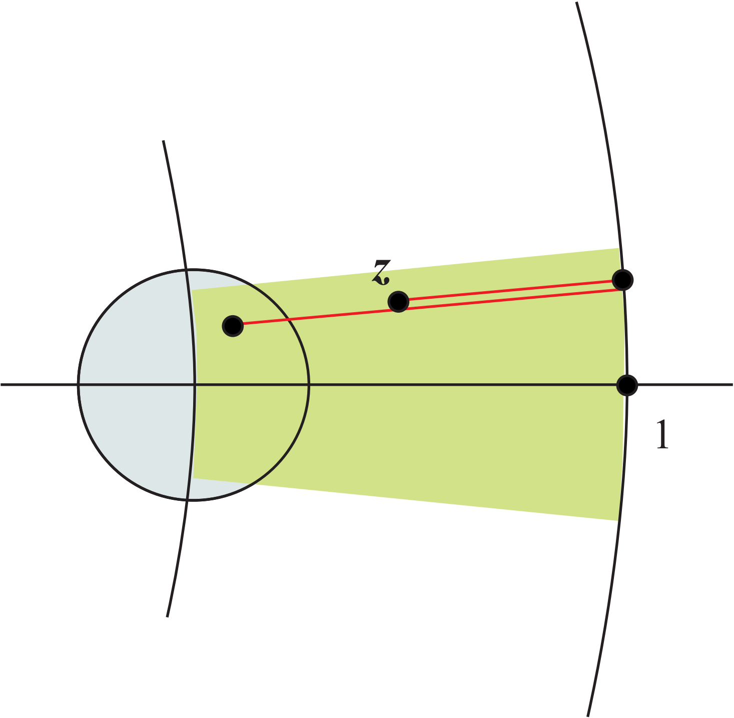
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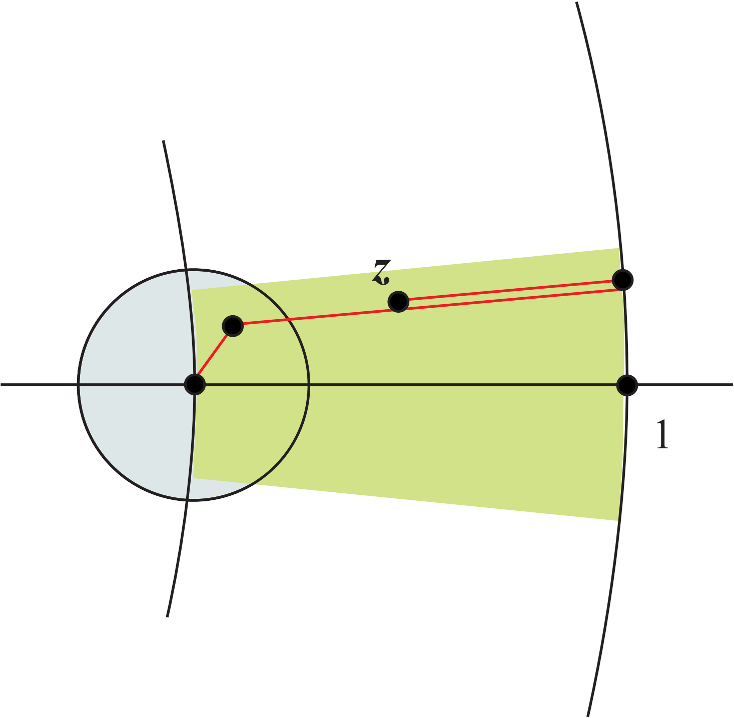
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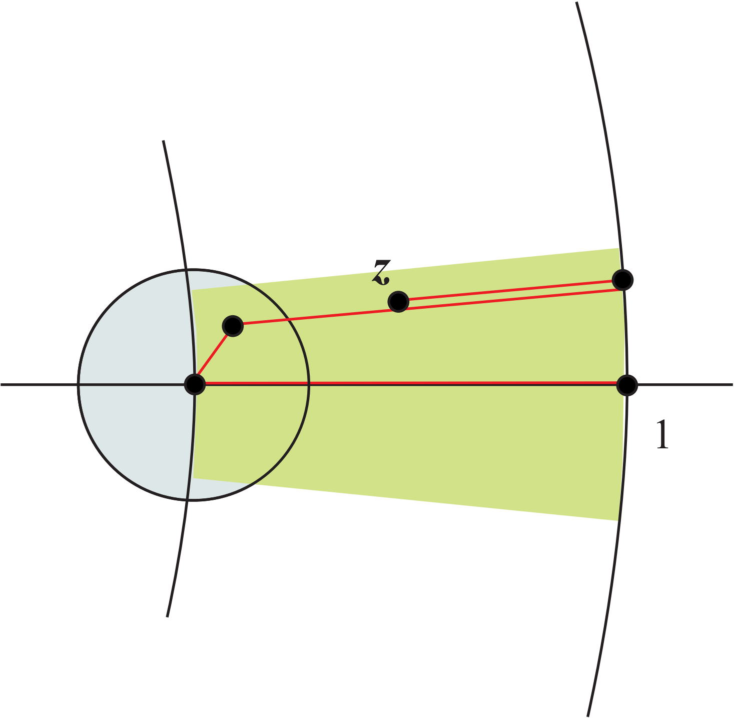
z
1

The Dilog Identity: $z = 1$

- Euler's identity says

$$\mathrm{Li}_2(-1/z) + \mathrm{Li}_2(-z) + \frac{1}{2}(\ln z)^2 = C$$

- This can be established for all z in $\mathbb{C} \setminus (-\infty, 0]$ as before, because in that domain $\mathrm{Li}_2(-1/z)$, $\mathrm{Li}_2(-z)$, and $\ln z$ are all differentiable.
- In particular, the identity holds for $z = 1$
- This shows (as before) that $C = 2\mathrm{Li}_2(-1)$
- Previous slide shows $\mathrm{Li}_2(-1) = \sum(-1)^k/k^2 = -S/2$

The Dilog Identity: $z = -1$

- Direct substitution invalid – identity doesn't hold at -1
- Consider sequence $z_k = e^{i\theta_k}$ approaching -1 along the unit circle counter-clockwise, with θ_k increasing to π
- Then $-1/z_k$ and $-z_k$ both approach 1 along the unit circle as well
- By continuity of Li_2 in U, $\mathrm{Li}_2(-1/z_k)$ and $\mathrm{Li}_2(-z_k)$ each approach $\mathrm{Li}_2(1) = S$
- $\ln z_k = i\theta_k \to i\pi$
- $\mathrm{Li}_2(-1/z_k) + \mathrm{Li}_2(-z_k) + (\ln z_k)^2/2 = -S$ for all k
- Take limits

Part III

The historical Question:

Did Euler know this proof?

History of Basel Problem

- Best source: "Euler's Solution of the Basel Problem - The Longer Story" in *Euler at 300: An Appreciation*, MAA, 2007, pp 105-117
- More sources: Euler Archive (on web), Dunham's *Master of Us All*, Varadarajan article in October 2007 AMS Bulletin
- Euler gave several proofs, noteworthy for their novelty, creativity, and genius
- Some of these proofs not completely rigorous by today's standards: unjustified manipulations of infinite series and products

Timeline

1730,1738 *De summatione innumerabilium progressionum* (E20): estimates S to 6 decimal places; derives identity $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln(x)\ln(1-x) = S$; shows $\mathrm{Li}_2(1/2) = S/2 - (\ln 2)^2/2$

1735, 1740 *De summis serierum reciprocarum* (E41): Three derivations of $S = \pi^2/6$. Some questionable steps.

1741, 1743 *Demonstration de la somme de cette suite 1 + 1/4 + 1/9 + 1/16 + etc.* (E63): Fourth proof $S = \pi^2/6$, using *only elementary calculus tools, Taylor series and integration by parts* (Sandifer). This proof as beyond criticism. Euler probably now considers his result as settled fact.

1768 *Institutionem Integralis* (E342, E366, E385): three volumes on integral calculus. Lewin says $\mathrm{Li}_2(z)$ is discussed here (somewhere).

1779, 1811 *De summatione serierum in hac forma contentarum* $a/1 + a^2/4 + a^3/9 + a^4/16 + a^5/25 + a^6/36 +$ *etc.* (E736). This paper presents the dilog identity used above. Note Euler's age (72). I don't know if this is the first publication of the identity.

DE SUMMATIONE SERIERUM

IN HAC FORMA CONTENTARUM:

$$\frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{10} + \frac{a^5}{25} + \frac{a^6}{36} + \text{etc.}$$

AUCTORE

L. EULERO.

Conventui exhibita die 31 Maji 1779.

§. 1. Ex iis quae olim primus de summatione potestatum reciprocarum in medium attuli, duo tantum casus derivari possunt, quibus summam seriei hic propositae assignare licet: alter scilicet quo $a = 1$, ubi ostendi hujus seriei: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} +$ etc. summam esse $= \frac{\pi\pi}{6}$, denotante π peripheriam circuli, cujus diameter $= 1$; alter vero casus est quo $a = -1$; tum enim mutatis signis

How (not) to Translate Latin

1. Scan document to create pdf file
2. Use OCR software to convert scanned image to character data
3. Hand edit the output to correctly reproduce the original text
4. Use on-line automated translation software
5. Using your fine understanding of what the math is saying together with the automatic translation of the basic Latin vocabulary, deduce the correct translation

Example

Corrected OCR output: Ex iis, quae olim primus de summatione potestatum reciprocarum in medium attuli, duo tantum casus derivari possunt, quibus summam seriei hic propositae assignare licet: alter scilicet quo a=1 ubi ostendi huius seriei

$$1 + 1/4 + 1/9 + \text{etc.}$$

summam esse = pp/6, denotante p peripheriam circuli, cuius diameter = 1;

Autotranslation (with corrected math): Out of iis , which at that time chief about summatione power reverberating upon mid attuli , two only a falling derivari to be able, by which the highest part sequence this proposal assignare it is allowed: the second rightly from a=1 when show of this sequence 1 + 1/4 + 1/9 etc. the highest part to be $\pi^2/6$ denotante π peripheriam round, of whom diameter = 1; the second in truth a falling is from a = -1; at that time in fact change signal of this sequence 1 - 1/4 + 1/9 etc. the highest part is $\pi^2/12$. ... Worthless in truth past these a falling ullas till then it is agreed, from the highest part assignare unimpeded.

Speculations

- Unlikely that Euler thought of using the dilog identity to derive the value of S
- Assuming E736(1779) is his first discovery of the identity, the timing seems wrong: he was far past the period when he was concerned about providing proofs that $S = \pi^2/6$.
- In E736 he seems content to use $S = \pi^2/6$ as an earlier result to determine other things about $\mathrm{Li}_2(z)$.
- I don't see him giving the above proof of $S = \pi^2/6$ in E376.
- He might have done so in another publication, but I have no evidence of that either way.
- Given earlier work with dilog, it is at least possible that he knew the identity used here far earlier, and might have realized it provided an alternate way to establish $S = \pi^2/6$
- Historical research is continuing with Mark McKinzie, St. John Fisher College and Erik Tou, The Euler Archive and Carthage College