Euler, Dilog, and the Basel Problem

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The Basel Problem

- Determine the exact value of $\sum_{k=1}^{\infty} \frac{1}{k^2}$
- Well known history: Euler's solution $= \pi^2/6$
- Approximately Avogadro's number of proofs now known
- This talk: standard power series methods lead to a simple proof, using the *dilog* function
- Euler again plays a central role
- Did he know this proof?

A Calc 2 Power Series Method

- Example: Determine $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

- Introduce a power series $f(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k}$

- We want $f(-1)$; we know $f(0) = 0$

- $f'(x) = \displaystyle\sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}$

- Therefore $f(x) = \displaystyle\int_0^x \frac{1}{1-t}\, dt = -\ln(1-x)$

- Conclusion: $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{k} = f(-1) = -\ln 2$

Power Series and the Basel Problem

- Want $\sum_{k=1}^{\infty} \frac{1}{k^2}$ so consider $f(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^2}$
- Differentiate and multiply by $x : xf'(x) = \sum_{k=1}^{\infty} \frac{x^k}{k}$
- Recognize this series as $-\ln(1-x)$
- $f'(x) = -\frac{\ln(1-x)}{x}$
- Constant of integration determined by $f(0) = 0$
- We want to find $f(1) = \int_0^1 -\frac{\ln(1-t)}{t}\, dt$

Evaluating the Integral

- Want $\displaystyle\int_0^1 -\frac{\ln(1-t)}{t}\,dt$
- Calc 2 methods of antidifferentiation inadequate
- Next: an indirect method using properties (identities) of the function

$$\mathrm{Li}_2(x) \equiv \sum_1^{\infty} \frac{x^k}{k^2} = -\int_0^x \frac{\ln(1-t)}{t}\,dt$$

- Note $\mathrm{Li}_2'(x) = -\dfrac{\ln(1-x)}{x}$

A Dilog Identity

- $\mathrm{Li}_2(-1/x) + \mathrm{Li}_2(-x) + \frac{1}{2}(\ln x)^2 = C$ (constant)
- Proof: Show the derivative of the left side is 0
- Recall $\mathrm{Li}_2'(x) = \dfrac{-\ln(1-x)}{x}$
- Set $x = 1 : C = 2\mathrm{Li}_2(-1)$
- Recall $\mathrm{Li}_2(x) = \displaystyle\sum_{k=1}^{\infty} \frac{x^k}{k^2}$
- $C = 2(-1 + 1/4 - 1/9 + 1/16 - \cdots)$

Odd and Even Terms

- Let $S = \sum 1/k^2$
- Even terms: $\sum 1/(2k)^2 = (1/4) \sum 1/k^2 = S/4$
- Odd terms: $S - S/4 = 3S/4$
- $\mathrm{Li}_2(-1) = -1 + 1/4 - 1/9 + 1/16 - \cdots =$ Evens - Odds $= -S/2$
- $\mathrm{Li}_2(-1/x) + \mathrm{Li}_2(-x) + \frac{1}{2}(\ln x)^2 = 2\mathrm{Li}_2(-1) = -S$

Play It Again, Sam

- $\mathrm{Li}_2(-1/x) + \mathrm{Li}_2(-x) + \frac{1}{2}(\ln x)^2 = 2\mathrm{Li}_2(-1) = -S$
- Set $x = -1$
- $2\mathrm{Li}_2(1) + \frac{1}{2}(\ln -1)^2 = -S$
- $2S + \frac{1}{2}(\ln -1)^2 = -S$
- $3S = -\frac{1}{2}(\ln -1)^2$
- $S = \frac{-(\ln -1)^2}{6}$

Finishing the Proof

- $S = \dfrac{-(\ln -1)^2}{6}$
- $-1 = e^{\pi i} \Rightarrow \ln -1 = \pi i$
  (note but ignore for now multi-value issues)
- $-(\ln -1)^2 = -(\pi^2 i^2) = \pi^2$
- $S = \dfrac{\pi^2}{6}$

Comments

- Two key ideas make the proof work – the dilog identity and $e^{\pi i} = -1$ – and Euler gave us BOTH!

- The argument given is in the Euler style – lots of creative formal manipulation, and ...

- ... complete rigor demands a bit more care: convergence, defining functions and integration for complex variables

- Historical Question: Was Euler aware of this method for solving the Basel problem?

- The above proof is given by Lewin in his book *Polylogarithms and Associated Functions*, North Holland, 1981. He doesn't claim to have invented the proof, but gives no attribution for it.

The historical Question:

Did Euler know this proof?

Timeline

1730,1738 *De summatione innumerabilium progressionum* (E20): estimates S to 6 decimal places; derives identity $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln(x)\ln(1-x) = S$; shows $\mathrm{Li}_2(1/2) = S/2 - (\ln 2)^2/2$

1735, 1740 *De summis serierum reciprocarum* (E41): Three derivations of $S = \pi^2/6$. Some questionable steps.

1741, 1743 *Demonstration de la somme de cette suite 1 + 1/4 + 1/9 + 1/16 + etc.* (E63): Fourth proof $S = \pi^2/6$, using *only elementary calculus tools, Taylor series and integration by parts* (Sandifer). This proof is beyond criticism. Euler presumably now considers his result as settled fact.

1768 *Institutionem Integralis* (E342, E366, E385): three volumes on integral calculus. Lewin says $\mathrm{Li}_2(z)$ is discussed here (somewhere).

1779, 1811 *De summatione serierum in hac forma contentarum* $a/1 + a^2/4 + a^3/9 + a^4/16 + a^5/25 + a^6/36+$ *etc.* (E736). This paper presents the dilog identity we used to derive $\pi^2/6$. Note Euler's age (72). I don't know if this is the first publication of the identity.

DE SUMMATIONE SERIERUM

IN HAC FORMA CONTENTARUM:

$$\frac{a}{1} + \frac{a^2}{4} + \frac{a^3}{9} + \frac{a^4}{10} + \frac{a^5}{25} + \frac{a^6}{36} + \text{etc.}$$

AUCTORE

L. EULERO.

Conventui exhibita die 31 Maji 1779.

§. 1. Ex iis quae olim primus de summatione potestatum reciprocarum in medium attuli, duo tantum casus derivari possunt, quibus summam seriei hic propositae assignare licet: alter scilicet quo $a = 1$, ubi ostendi hujus seriei: $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} +$ etc. summam esse $= \frac{\pi\pi}{6}$, denotante π peripheriam circuli, cujus diameter $= 1$; alter vero casus est quo $a = -1$; tum enim, mutatis signis, hujus seriei: $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} -$ etc. summa est $= \frac{\pi\pi}{12}$.

What did he know, when did he know it?

- 1730: $\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) + \ln(x)\ln(1-x) = C$ (and $C = \mathrm{Li}_2(1)$)
- 1779: $\mathrm{Li}_2(-1/x) + \mathrm{Li}_2(-x) + \frac{1}{2}(\ln x)^2 = B$ (constant) and $B = 2\mathrm{Li}_2(-1)$
- When did he first discover the second identity?

Methods of 1730 Paper (E20)

- Source: Varadarajan email about his Bulletin Paper

- Using V's notation

- $T(\alpha) = \dfrac{1}{\alpha} + \dfrac{1}{2(\alpha+1)} + \dfrac{1}{3(\alpha+2)} + \cdots$

- $T = S$ when $\alpha = 1$

- Euler obtains for $0 < u < 1$

$$T(\alpha) = \sum_{r=0}^{\infty} \frac{u^{r+\alpha}}{(r+\alpha)(r+1)} + \sum_{r=0}^{\infty} \left(\begin{array}{c} (r+1-\alpha) \\ r \end{array} \right) \frac{(1-u)^{r+1}}{(r+1)^2}$$

$$-\log(1-u) \sum_{r=0}^{\infty} \left(\begin{array}{c} (r+1-\alpha) \\ r \end{array} \right) \frac{(1-u)^{r+1}}{(r+1)}$$

- Take $\alpha = 1$ to derive 1730 dilog identity

Methods of 1779 Paper (E736)

- Let $p = \int \frac{\ln y}{x}\, dx$ and $q = \int \frac{\ln x}{y}\, dy$
- Then $p + q = \ln x \cdot \ln y + C$
- Essentially integration by parts
- It's easy to follow Euler's steps in the original paper

PROBLEMA 2

Si fuerit $x - y = 1$, *binas illas formulas*

$$p = \int \frac{dx}{x} ly \quad \text{et} \quad q = \int \frac{dy}{y} lx$$

in series resolvere, ita ut hinc prodeat

$$p + q = lx \cdot ly + C.$$

SOLUTIO

5. Cum hic sit $y = x - 1$, erit

$$ly = l(x-1) = lx + l\left(1 - \frac{1}{x}\right) = lx - \frac{1}{x} - \frac{1}{2xx} - \frac{1}{3x^3} - \frac{1}{4x^4} - \text{etc.}$$

hincque

$$p = \int \frac{\partial x}{x} ly = \frac{1}{2}(lx)^2 + \frac{1}{x} + \frac{1}{4x^2} + \frac{1}{9x^3} + \frac{1}{16x^4} + \text{etc.}$$

Deinde ob $x = 1 + y$ erit

$$lx = \frac{y}{1} - \frac{yy}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \text{etc.}$$

ideoque

$$q = \int \frac{\partial y}{y} lx = \frac{y}{1} - \frac{y^2}{4} + \frac{y^3}{9} - \frac{y^4}{16} + \text{etc.}$$

Identity Storm

- By similar methods Euler produces many identities
- He looks at a variety of equations linking x and y: $x + y = 1$, $x - y = 1$, $xy + x + y = c$
- Evaluates constants of integration by artful specification of x
- Ends with 4 theorems: the two identities cited here and two additional similar identities

Euler and Daniel Bernoulli

- Early 1742: Euler and Daniel Bernoulli discuss power series manipulations leading to $1 + \frac{1}{4} + \frac{1}{9} + \cdots = \int \frac{1}{x} \ln \frac{1}{1-x} dx$;

- Euler wonders whether the method could apply to series in which exponents do not form an arithmetic progression

- Euler poses challenge to evaluate $a + a^4 + a^9 + a^{16} + \cdots$ by similar means; a successful solution *would shed new light on mathematics*

Euler and Christian Goldbach

- Aug 1742: Euler cites $x + \frac{x^2}{4} + \frac{x^3}{9} + \ldots$ as an interesting generalization of the Basil Problem; value only known for $x = 1, -1, 1/2$

- Euler closes with the following (*which could be of great use*):

  If $s = \sum_{k=0}^{\infty} \frac{a^k}{kn+1}$, then

$$\frac{s^2}{2} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{a^k}{kn+2} \frac{1}{jn+1}$$

- Jan 1743: Euler gives proof of this identity; shows how to apply it to compute $Li_2(1/2)$.

The Euler-Goldbach correspondence edited by Fuss (1843); we anticipate fresh insights when Opera Omnia, Ser 4, vol 4 appears... (2011)

Euler and D'Alembert on the logarithm function

- Bernoulli's formula for area of circular sector (radius a, angle having sine y, cosine x): area $= \frac{a^2}{4\sqrt{-1}} \ln \frac{x+y\sqrt{-1}}{x-y\sqrt{-1}}$ (Bernoulli, 1702).
- 1728: Euler notes that when $x = 0$ we find that $\ln(-1) = \pi\sqrt{-1}$
- Dec 1746 (letter to D'Alembert): "I believe that I have proved that $[\ln(-1)]$ is imaginary and that it is $= \pi(1 \pm 2n)\sqrt{-1} \ldots$"
- D'Alembert replies with a host of objections... e.g., since $-1 = 1/-1$, it follows that $\ln(-1) = \ln(1) - \ln(-1)$, so that $2\ln(-1) = \ln(1) = 0$.
- April 1747: Euler notes that essentially the same argument would show that $\ln\sqrt{-1} = 0$, which contradicts Bernoulli's area formula

Euler and D'Alembert on the logarithm function

- August 1747: Euler submits his paper on logarithms of negative and imaginary quantities [E-807] "where I believe I have put this matter to rest; at least for my part, I have not the least difficulty with it, whereas I had previously been extremely perplexed."

[Source: R. Bradley, "Euler, D'Alembert and the Logarithm Function", *Leonhard Euler: Life, Work, and Legacy* (2007: Elsevier), pp 255 – 277]

Conclusion

- Elementary (calc2) series methods offer obvious approach to solving the Basil problem
- It succeeds thanks to two key identities due to Euler
- Did Euler know this proof?
- Surely when he wrote E736 Euler would have seen that a minor rearrangement would prove (yet again) that $\sum 1/k^2 = \pi^2/6$, but might not have considered this noteworthy
- If Euler was aware earlier, say prior to 1747, he might have been reluctant to depend on evaluating $\ln -1$.
- More may be revealed by the correspondence soon to be released.

References

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3. Euler Papers, especially E20 and E736: Euler Archive (on web)
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5. Lewin, *Polylogarithms and Associated Functions*, North Holland, 1981
6. R. Bradley, "Euler, D'Alembert and the Logarithm Function", *Leonhard Euler: Life, Work, and Legacy* (2007: Elsevier), pp 255 – 277.