The Fibonacci Numbers – Exposed

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Super-sequence OR Mild Mannered Recursion?

- Fibonacci numbers are famous for amazing properties
- Gushy Koshy: One of . . . *two shining stars in the vast array of integer sequences*
- The Planet of Two-term Recurrences: super sequences all

Outline

- Famous Fibonacci Facts
- Generalized Fibonacci and Lucas Numbers
- Difference Operators
- Binet Formulas
- Matrix Methods
- Pythagorean Triples
- GCD Property

Fibonacci and Lucas Numbers

- $F_n : 0, 1, 1, 2, 3, 5, 8, \cdots$
- $L_n : 2, 1, 3, 4, 7, 11, 18 \cdots$
- Each term is the sum of two preceding terms
- $F_{n+2} = F_{n+1} + F_n$
- $L_{n+2} = L_{n+1} + L_n$

Five Fibonacci Facts

1. $\sum_{1}^{n} F_i^2 = F_n F_{n+1}$.

2. $L_{n+1} = F_{n+2} + F_n$

3. $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$ where α, β given by $(1 \pm \sqrt{5})/2$

4. $F_{n+1}/F_n \to \alpha$ as $n \to \infty$.

5. $\sum_{1}^{n} F_i = F_{n+2} - 1$.

Another Five Fibonacci Facts

6. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}$

7. $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$.

8. $F_n = F_m F_{n-m+1} + F_{m-1} F_{n-m}$.

9. If w, x, y, z are four consecutive Fibonacci numbers, then $(wz, 2xy, yz - wx)$ is a Pythagorean triple

10. $\mathrm{GCD}(F_m, F_n) = F_{\mathrm{GCD}(m,n)}$

(a,b) Fibonacci & Lucas Numbers

- Consider fixed constants a and b
- Consider sequences A_n satisfying $A_{n+2} = aA_{n+1} + bA_n$.
- Set of all such sequences is $\mathcal{R}(a, b)$.
- (a, b)-Fibonacci numbers: $F \in \mathcal{R}(a, b)$ starting $0, 1, a, \ldots$.
- (a, b)-Lucas numbers: $L \in \mathcal{R}(a, b)$ starting $2, a, a^2 + 2b, \ldots$.

Examples

- $\mathcal{R}(11,-10)$. $F = 0, 1, 11, 111, 1111, \ldots$, $L = 2, 11, 101, 1001, 10001, \ldots$.
- $\mathcal{R}(2,-1)$. $F = 0, 1, 2, 3, 4, \ldots$, $L = 2, 2, 2, 2, \ldots$.
- $\mathcal{R}(1,-1)$. $F = 0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots$,
  $L = 2, 1, -1, -2, -1, 1, 2, 1, -1, \ldots$.
- $\mathcal{R}(3,-1)$. $F = 0, 1, 3, 8, 21, \ldots$, $L = 2, 3, 7, 18, \ldots$.

Sample Generalization

- Fibonacci Fact: $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$
- In $\mathcal{R}(a,b)$:$b^n F_0^2 + b^{n-1} F_1^2 + \cdots + b F_{n-1}^2 + F_n^2 = \dfrac{F_n F_{n+1}}{a}$
- Proof: Direct Induction

Operators on $\mathcal{R}(a, b)$

- Left shift $\Lambda : (A_0, A_1, \cdots) \to (A_1, A_2, \cdots)$
- $A \in \mathcal{R}(a, b)$ iff $(\Lambda^2 - a\Lambda - b)A = 0$
- Difference $\Delta : (A_0, A_1, \cdots) \to (A_1 - A_0, A_2 - A_1, \cdots)$
- Note $\Delta = \Lambda - 1$
- k term sum $\Sigma_k = 1 + \Lambda + \cdots + \Lambda^{k-1}$
- Running sum $\Sigma : (A_0, A_1, \cdots) \to (A_0, A_0 + A_1, A_0 + A_1 + A_2, \cdots)$
- k skip $\Omega_k : (A_0, A_1, \cdots) \to (A_0, A_k, A_{2k}, \cdots)$

Properties of $\mathcal{R}(a, b)$

- $\mathcal{R}(a, b)$ = null space of $\Lambda^2 - a\Lambda - b$ (so its a linear space)
- Dimension of $\mathcal{R}(a, b)$ is 2
- $\mathcal{R}(a, b)$ preserved by any operator which commutes with Λ, including Δ, Σ_k, and any polynomial in Λ
- F and L are independent elements, so form a basis for $\mathcal{R}(a, b)$.
- Any element of $\mathcal{R}(a, b)$ can be expressed as a combination of F and L
- All differences and k term sums of elements of $\mathcal{R}(a, b)$ can be expressed as a combination of F and L

A Proof

If $\Psi\Lambda = \Lambda\Psi$ then $\Psi : \mathcal{R}(a, b) \to \mathcal{R}(a, b)$.
Proof: Let $A \in \mathcal{R}(a, b)$, and consider $\Psi(A)$. We have

$$\begin{aligned}
(\Lambda^2 - a\Lambda - b)\Psi(A) &= \Psi(\Lambda^2 - a\Lambda - b)A \\
&= \Psi(0) \\
&= 0
\end{aligned}$$

Thus, $\Psi(A) \in \mathcal{R}(a, b)$.

More Properties of $\mathcal{R}(a, b)$

- Natural basis: $E = 1, 0, b, \ldots,$ and $F = 0, 1, a, \ldots$
- $A = A_0 E + A_1 F$ for all $A \in \mathcal{R}(a, b)$
- $\Lambda E = bF$
- $A_n = bA_0 F_{n-1} + A_1 F_n$
- $L_n = 2bF_{n-1} + aF_n = bF_{n-1} + F_{n+1}$

Binet Formulas

- Special elements of $\mathcal{R}(a, b)$: geometric progressions
- $\Lambda(\{\lambda^n\}) = \{\lambda^{n+1}\} = \lambda \cdot \{\lambda^n\}$
- Requirement for $\lambda : \lambda^2 - a\lambda - b = 0$
- Usual case: two distinct solutions $\Rightarrow \mathcal{R}(a, b)$ has basis $\{\lambda^n, \mu^n\}$
- Every A in $\mathcal{R}(a, b)$ can be expressed as a sum of two geometric progressions. In particular $F_n = (\lambda^n - \mu^n)/(\lambda - \mu)$ and $L_n = \lambda^n + \mu^n$
- Repeated roots handled as a separate case.

Binet Formula Consequences

- $\frac{A_{n+1}}{A_n} = \frac{c_\lambda \lambda^{n+1} + c_\mu \mu^{n+1}}{c_\lambda \lambda^n + c_\mu \mu^n} \to \lambda \quad \text{as} \quad n \to \infty.$

- $\Omega_k A_n = c_\lambda \lambda^{kn} + c_\mu \mu^{kn}$. This is in $\mathcal{R}(a', b')$ – the combinations of geometric progressions based on λ^k and μ^k

- In fact, $a' = L_k^{(a,b)}$ and $b' = -(-b)^k$ and

$$\begin{aligned} \Omega_k F^{(a,b)} &= F_k^{(a,b)} \cdot F^{(a',b')} \\ \Omega_k L^{(a,b)} &= L^{(a',b')} \end{aligned}$$

Another Binet Consequence

$$\begin{aligned} A_n &= c_\lambda \lambda^n + c_\mu \mu^n \\ \Sigma A_n &= c_\lambda \frac{\lambda^{n+1}-1}{\lambda-1} + c_\mu \frac{\mu^{n+1}-1}{\mu-1} \\ &= \frac{c_\lambda \lambda}{\lambda-1}\lambda^n + \frac{c_\mu \mu}{\mu-1}\mu^n - \left(\frac{c_\lambda}{\lambda-1} + \frac{c_\mu}{\mu-1}\right) \end{aligned}$$

Conclusion: if $A \in \mathcal{R}(a,b)$ then ΣA is a constant plus an element of $\mathcal{R}(a,b)$.

Special Case:

$$\Sigma F_n = \frac{1}{a+b-1}(F_{n+1} + bF_n - 1)$$

Matrix Formulation

- Represent elements of $\mathcal{R}(a,b)$ as coordinate vectors WRT E and F : $[A] = [A_0 \;\; A_1]^T$
- Now Λ is represented by a matrix M
- $M = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}$
- $[A_1 \;\; A_2]^T = [\Lambda(A)] = M[A] = M[A_0 \;\; A_1]^T$
- $[A_k \;\; A_{k+1}]^T = [\Lambda^k(A)] = M^k[A] = M^k[A_0 \;\; A_1]^T$

Matrix Rep (cont.)

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}^n \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = A_n$$

$$\begin{aligned} M^n &= M^n I \\ &= \left[M^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad M^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] \\ &= \begin{bmatrix} E_n & F_n \\ E_{n+1} & F_{n+1} \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}^n &= \begin{bmatrix} bF_{n-1} & F_n \\ bF_n & F_{n+1} \end{bmatrix} \end{aligned}$$

Consequences of Matrix Rep

$$\begin{aligned}
\det(M^n) &= (\det M)^n \\
b(F_{n-1}F_{n+1} - F_n^2) &= (-b)^n
\end{aligned}$$

$$\begin{aligned}
M^n &= M^m M^{n-m} \\
\begin{bmatrix} bF_{n-1} & F_n \\ bF_n & F_{n+1} \end{bmatrix} &= \begin{bmatrix} bF_{m-1} & F_m \\ bF_m & F_{m+1} \end{bmatrix} \begin{bmatrix} bF_{n-m-1} & F_{n-m} \\ bF_{n-m} & F_{n-m+1} \end{bmatrix} \\
F_n &= F_m F_{n-m+1} + bF_{m-1}F_{n-m}
\end{aligned}$$

b-Pythagorean Triples

- (X, Y, Z) is a b-Pythagorean triple iff $X^2 + bY^2 = Z^2$
- For any u, v, $(v^2 - bu^2, 2uv, v^2 + bu^2)$ is a b-Pythagorean triple
- Define $c = b/a$, and $d = c - a$. Then for any consective terms w, x, y, and z of a sequence $A \in \mathcal{R}(a, b)$ we can construct a b-Pythagorean triple as $(cwz - dxy, 2xy, xz + bwy)$

GCD Preliminaries

- Consider positive integer values of a and b
- Any element of $\mathcal{R}(a, b)$ with integer initial values is an integer sequence. In particular, F is an integer sequence.
- Assume a and b are relatively prime
- In $\mathcal{R}(a, b)$ F_h and F_k are relatively prime whenever h and k are

GCD Property

The GCD of F_m and F_n is F_k where k is the GCD of m and n.

Proof: Since $k = \mathrm{GCD}(m, n)$ we can write $m = rk$ and $n = sk$ for relatively prime integers r and s. We consider the sequence $F_k, F_{2k}, F_{3k}, \ldots$. This is the k skip of $F^{(a,b)}$ which we know is given by $F_k \cdot F^{(a',b')}$. So clearly F_k is a common divisor of F_{rk} and F_{sk}. If we devide the terms by this common divisor, we are left with $F_r^{(a',b')}$ and $F_s^{(a',b')}$. These are relatively prime because r and s are. This makes F_k the *greatest* common divisor.