The Fibonacci Numbers—Exposed

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Among numerical sequences, the Fibonacci numbers F_n have achieved a kind of celebrity status. Indeed, Koshy gushingly refers to them as one of the "two shining stars in the vast array of integer sequences" [**16**, p. xi]. The second of Koshy's "shining stars" is the *Lucas* numbers, a close relative of the Fibonacci numbers, about which we will say more below. The Fibonacci numbers are famous for possessing wonderful and amazing properties. Some are well known. For example, the sums and differences of Fibonacci numbers are Fibonacci numbers, and the ratios of Fibonacci numbers converge to the golden mean. Others are less familiar. Did you know that any four consecutive Fibonacci numbers can be combined to form a Pythagorean triple? Or how about this: The greatest common divisor of two Fibonacci numbers is another Fibonacci number. More precisely, the gcd of F_n and F_m is F_k, where k is the gcd of n and m.

With such fabulous properties, it is no wonder that the Fibonacci numbers stand out as a kind of super sequence. But what if it is not such a special sequence after all? What if it is only a rather pedestrian sample from an entire race of super sequences? In this case, the home world is the planet of two term recurrences. As we shall show, its inhabitants are all just about as amazing as the Fibonacci sequence.

The purpose of this paper is to demonstrate that many of the properties of the Fibonacci numbers can be stated and proved for a much more general class of sequences, namely, second-order recurrences. We shall begin by reviewing a selection of the properties that made Fibonacci numbers famous. Then there will be a survey of second-order recurrences, as well as general tools for studying these recurrences. A number of the properties of the Fibonacci numbers will be seen to arise simply and naturally as the tools are presented. Finally, we will see that Fibonacci connections to Pythagorean triples and the gcd function also generalize in a natural way.

Famous Fibonacci properties

The Fibonacci numbers F_n are the terms of the sequence $0, 1, 1, 2, 3, 5, \ldots$ wherein each term is the sum of the two preceding terms, and we get things started with 0 and 1 as F_0 and F_1. You cannot go very far in the lore of Fibonacci numbers without encountering the companion sequence of Lucas numbers L_n, which follows the same recursive pattern as the Fibonacci numbers, but begins with $L_0 = 2$ and $L_1 = 1$. The first several Lucas numbers are therefore 2, 1, 3, 4, 7.

Regarding the origins of the subject, Koshy has this to say:

> The sequence was given its name in May of 1876 by the outstanding French mathematician François Edouard Anatole Lucas, who had originally called it "the series of Lamé," after the French mathematician Gabriel Lamé [**16**, p. 5].

Although Lucas contributed a great deal to the study of the Fibonacci numbers, he was by no means alone, as a perusal of Dickson [**4**, Chapter XVII] reveals. In fact, just about all the results presented here were first published in the nineteenth century. In particular, in his foundational paper [**17**], Lucas, himself, investigated the generalizations that interest us. These are sequences A_n defined by a recursive rule $A_{n+2} = aA_{n+1} + bA_n$ where a and b are fixed constants. We refer to such a sequence as a *two-term recurrence*.

The popular lore of the Fibonacci numbers seems not to include these generalizations, however. As a case in point, Koshy [**16**] has devoted nearly 700 pages to the properties of Fibonacci and Lucas numbers, with scarcely a mention of general two-term recurrences. Similar, but less encyclopedic sources are Hoggatt [**9**], Honsberger [**11**, Chapter 8], and Vajda [**21**]. There has been a bit more attention paid to so-called *generalized Fibonacci numbers*, A_n, which satisfy the same recursive formula $A_{n+2} = A_{n+1} + A_n$, but starting with arbitrary initial values A_0 and A_1, particularly by Horadam (see for example Horadam [**12**], Walton and Horadam [**22**], as well as Koshy [**16**, Chapter 7]). Horadam also investigated the same sort of sequences we consider, but he focused on different aspects from those presented here [**14, 15**]. In [**14**] he includes our Examples 3 and 7, with an attribution to Lucas's 1891 *Théorie des Nombres*. With Shannon, Horadam also studied Pythagorean triples, and their paper [**20**] goes far beyond the connection to Fibonacci numbers considered here. Among more recent references, Bressoud [**3**, chapter 12] discusses the application of generalized Fibonacci sequences to primality testing, while Hilton and Pedersen [**8**] present some of the same results that we do. However, none of these references share our general point of emphasis, that in many cases, properties commonly perceived as unique to the Fibonacci numbers, are actually shared by large classes of sequences.

It would be impossible to make this point here in regard to all known Fibonacci properties, as Koshy's tome attests. We content ourselves with a small sample, listed below. We have included page references from Koshy [**16**].

Sum of squares $\sum_1^n F_i^2 = F_n F_{n+1}$. (Page 77.)

Lucas-Fibonacci connection $L_{n+1} = F_{n+2} + F_n$. (Page 80.)

Binet formulas The Fibonacci and Lucas numbers are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}.$$

(Page 79.)

Asymptotic behavior $F_{n+1}/F_n \to \alpha$ as $n \to \infty$. (Page 122.)

Running sum $\sum_1^n F_i = F_{n+2} - 1$. (Page 69.)

Matrix form We present a slightly permuted form of what generally appears in the literature. Our version is

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{bmatrix}.$$

(Page 363.)

Cassini's formula $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$. (Page 74)

Convolution property $F_n = F_m F_{n-m+1} + F_{m-1} F_{n-m}$. (Page 88, formula 6.)

Pythagorean triples If w, x, y, z are four consecutive Fibonacci numbers, then $(wz, 2xy, yz - wx)$ is a Pythagorean triple. That is, $(wz)^2 + (2xy)^2 = (yz - wx)^2$. (Page 91, formula 88.)

Greatest common divisor $\gcd(F_m, F_n) = F_{\gcd(m,n)}$. (Page 198.)

This is, as mentioned, just a sample of amazing properties of the Fibonacci and Lucas numbers. But they all generalize in a natural way to classes of two-term recurrences. In fact, several of the proofs arise quite simply as part of a general development of the recurrences. We proceed to that topic next.

Generalized Fibonacci and Lucas numbers

Let a and b be any real numbers. Define a sequence A_n as follows. Choose initial values A_0 and A_1. All succeeding terms are determined by

$$A_{n+2} = aA_{n+1} + bA_n. \tag{1}$$

For fixed a and b, we denote by $\mathcal{R}(a, b)$ the set of all such sequences. To avoid a trivial case, we will assume that $b \neq 0$.

In $\mathcal{R}(a, b)$, we define two distinguished elements. The first, F, has initial terms 0 and 1. In $\mathcal{R}(1, 1)$, F is thus the Fibonacci sequence. In the more general case, we refer to F as the (a, b)-Fibonacci sequence. Where no confusion will result, we will suppress the dependence on a and b. Thus, in every $\mathcal{R}(a, b)$, there is an element F that begins with 0 and 1, and this is the Fibonacci sequence for $\mathcal{R}(a, b)$.

Although F is the primordial sequence in $\mathcal{R}(a, b)$, there is another sequence L that is of considerable interest. It starts with $L_0 = 2$ and $L_1 = a$. As will soon be clear, L plays the same role in $\mathcal{R}(a, b)$ as the Lucas numbers play in $\mathcal{R}(1, 1)$. Accordingly, we refer to L as the (a, b)-Lucas sequence. For the most part, there will be only one a and b under consideration, and it will be clear from context which $\mathcal{R}(a, b)$ is the home for any particular mention of F or L. In the rare cases where some ambiguity might occur, we will use $F^{(a,b)}$ and $L^{(a,b)}$ to indicate the F and L sequences in $\mathcal{R}(a, b)$.

In the literature, what we are calling F and L have frequently been referred to as *Lucas sequences* (see Bressoud [**3**, chapter 12] and Weisstein [**23**, p. 1113]) and denoted by U and V, the notation adopted by Lucas in 1878 [**17**]. We prefer to use F and L to emphasize the idea that there are Fibonacci and Lucas sequences in each $\mathcal{R}(a, b)$, and that these sequences share many properties with the traditional F and L. In contrast, it has sometimes been the custom to attach the name *Lucas* to the L sequence for a particular $\mathcal{R}(a, b)$. For example, in $\mathcal{R}(2, 1)$, the elements of F have been referred to as *Pell numbers* and the elements of L as *Pell-Lucas* numbers [**23**, p. 1334].

Examples Of course, the most familiar example is $\mathcal{R}(1, 1)$, in which F and L are the famous Fibonacci and Lucas number sequences. But there are several other choices of a and b that lead to familiar examples.

Example 1: $\mathcal{R}(11, -10)$. The Fibonacci sequence in this family is $F = 0, 1, 11, 111, 1111, \ldots$ the sequence of repunits, and $L = 2, 11, 101, 1001, 10001, \ldots$. The initial 2, which at first seems out of place, can be viewed as the result of putting two 1s in the same position.

Example 2: $\mathcal{R}(2, -1)$. Here F is the sequence of whole numbers $0, 1, 2, 3, 4, \ldots$, and L is the constant sequence $2, 2, 2, \ldots$. More generally, $\mathcal{R}(2, -1)$ consists of all the arithmetic progressions.

Example 3: $\mathcal{R}(3, -2)$. $F = 0, 1, 3, 7, 15, 31, \ldots$ is the Mersenne sequence, and $L = 2, 3, 5, 9, 17, 33, \ldots$ is the Fermat sequence. These are just powers of 2 plus or minus 1.

Example 4: $\mathcal{R}(1, -1)$. $F = 0, 1, 1, 0, -1, -1, 0, 1, 1, \ldots$ and $L = 2, 1, -1, -2, -1, 1, 2, 1, -1, \ldots$. Both sequences repeat with period 6, as do all the elements of $\mathcal{R}(1, -1)$.

Example 5: $\mathcal{R}(3, -1)$. $F = 0, 1, 3, 8, 21, \ldots$ and $L = 2, 3, 7, 18, \ldots$. Do you recognize these? They are the *even-numbered* Fibonacci and Lucas numbers.

Example 6: $\mathcal{R}(4, 1)$. $F = 0, 1, 4, 17, 72, \ldots$ and $L = 2, 4, 18, 76, \ldots$. Here, L gives every third Lucas number, while F gives 1/2 of every third Fibonacci number.

Example 7: $\mathcal{R}(2, 1)$. $F = 0, 1, 2, 5, 12, 29, 70, \ldots$ and $L = 2, 2, 6, 14, 34, 82, \ldots$. These are the Pell sequences, mentioned earlier. In particular, for any n, $(x, y) = (F_{2n} + F_{2n-1}, F_{2n})$ gives a solution to Pell's Equation $x^2 - 2y^2 = 1$. This extends to the more general Pell equation, $x^2 - dy^2 = 1$, when $d = k^2 + 1$. Then, using the F sequence in $\mathcal{R}(2k, 1)$, we obtain solutions of the form $(x, y) = (kF_{2n} + F_{2n-1}, F_{2n})$. Actually, equations of this type first appeared in the Archimedean cattle problem, and were considered by the Indian mathematicians Brahmagupta and Bhaskara [**2**, p. 221]. Reportedly, Pell never worked on the equations that today bear his name. Instead, according to Weisstein [**23**], "while Fermat deserves the credit for being the first [European] to extensively study the equation, the erroneous attribution to Pell was perpetrated by none other than Euler."

Coincidentally, the even terms F_{2n} in $\mathcal{R}(a, 1)$ also appear in another generalized Fibonacci result, related to an identity discussed elsewhere in this issue of the MAGAZINE [**6**]. The original identity for normal Fibonacci numbers is

$$\arctan\left(\frac{1}{F_{2n}}\right) = \arctan\left(\frac{1}{F_{2n+1}}\right) + \arctan\left(\frac{1}{F_{2n+2}}\right).$$

For $F \in \mathcal{R}(a, 1)$ the corresponding result is

$$\arctan\left(\frac{1}{F_{2n}}\right) = \arctan\left(\frac{a}{F_{2n+1}}\right) + \arctan\left(\frac{1}{F_{2n+2}}\right).$$

The wonderful world of two-term recurrences

The Fibonacci and Lucas sequences are elements of $\mathcal{R}(1, 1)$, and many of their properties follow immediately from the recursive rule that each term is the sum of the two preceding terms. Similarly, it is often easy to establish corresponding properties for elements of $\mathcal{R}(a, b)$ directly from the fundamental identity (1). For example, in $\mathcal{R}(1, 1)$, the *Sum of Squares* identity is

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}.$$

The generalization of this to $\mathcal{R}(a, b)$ is

$$b^n F_0^2 + b^{n-1} F_1^2 + \cdots + bF_{n-1}^2 + F_n^2 = \frac{F_n F_{n+1}}{a}. \tag{2}$$

This can be proved quite easily using (1) and induction.

Many of the other famous properties can likewise be established by induction. But to provide more insight about these properties, we will develop some analytic methods, organized loosely into three general contexts. First, we can think of $\mathcal{R}(a, b)$ as a subset of $\mathbb{R}^\infty$, the real vector space of real sequences, and use the machinery of difference operators. Second, by deriving Binet formulas for elements of $\mathcal{R}(a, b)$, we obtain explicit representations as linear combinations of geometric progressions. Finally, there is a natural matrix formulation which is tremendously useful. We explore each of these contexts in turn.

Difference operators We will typically represent elements of $\mathbb{R}^\infty$ with uppercase roman letters, in the form

$$A = A_0, A_1, A_2, \dots.$$

There are three fundamental linear operators on $\mathbb{R}^\infty$ to consider. The first is the *left-shift*, Λ. For any real sequence $A = A_0, A_1, A_2, \dots$, the shifted sequence is $\Lambda A = A_1, A_2, A_3, \dots$.

This shift operator is a kind of discrete differential operator. Recurrences like (1) are also called difference equations. Expressed in terms of Λ, (1) becomes

$$(\Lambda^2 - a\Lambda - b)A = 0.$$

This is analogous to expressing a differential equation in terms of the differential operator, and there is a theory of difference equations that perfectly mirrors the theory of differential equations. Here, we have in mind linear constant coefficient differential and difference equations.

As one fruit of this parallel theory, we see at once that $\Lambda^2 - a\Lambda - b$ is a linear operator on $\mathbb{R}^\infty$, and that $\mathcal{R}(a, b)$ is its null space. This shows that $\mathcal{R}(a, b)$ is a subspace of $\mathbb{R}^\infty$. We will discuss another aspect of the parallel theories of difference and differential equation in the succeeding section on Binet formulas.

Note that any polynomial in Λ is a linear operator on $\mathbb{R}^\infty$, and that all of these operators commute. For example, the *forward difference* operator Δ, defined by $(\Delta A)_k = A_{k+1} - A_k$, is given by $\Delta = \Lambda - 1$. Similarly, consider the k-term sum, Σ_k, defined by $(\Sigma_k A)_n = A_n + A_{n+1} + \cdots + A_{n+k-1}$. To illustrate, $\Sigma_2(A)$ is the sequence $A_0 + A_1, A_1 + A_2, A_2 + A_3, \dots$. These sum operators can also be viewed as polynomials in Λ: $\Sigma_k = 1 + \Lambda + \Lambda^2 + \cdots + \Lambda^{k-1}$.

Because these operators commute with Λ, they are operators on $\mathcal{R}(a, b)$, as well. In general, if Ψ is an operator that commutes with Λ, we observe that Ψ also commutes with $\Lambda^2 - a\Lambda - b$. Thus, if $A \in \mathcal{R}(a, b)$, then $(\Lambda^2 - a\Lambda - b)\Psi A = \Psi(\Lambda^2 - a\Lambda - b)A = \Psi 0 = 0$. This shows that $\Psi A \in \mathcal{R}(a, b)$. In particular, $\mathcal{R}(a, b)$ is closed under differences and k-term sums.

This brings us to the second fundamental operator, the *cumulative sum* Σ. It is defined as follows: $\Sigma(A) = A_0, A_0 + A_1, A_0 + A_1 + A_2, \dots$. This is not expressible in terms of Λ, nor does it commute with Λ, in general. However, there is a simple relation connecting the two operators:

$$\Delta\Sigma = \Lambda. \tag{3}$$

This is a sort of discrete version of the fundamental theorem of calculus. In the opposite order, we have

$$(\Sigma\Delta A)_n = A_{n+1} - A_0,$$

a discrete version of the other form of the fundamental theorem. It is noteworthy that Leibniz worked with these sum and difference operators as a young student, and later identified this work as his inspiration for calculus (Edwards [**5**, p. 234]).

The final fundamental operator is the *k-skip*, Ω_k, which selects every kth element of a sequence. That is, $\Omega_k(A) = A_0, A_k, A_{2k}, A_{3k}, \ldots$. By combining these operators with powers of Λ, we can sample the terms of a sequence according to any arithmetic progression. For example,

$$\Omega_5 \Lambda^3 A = A_3, A_8, A_{13}, \ldots.$$

Using the context of operators and the linear space $\mathcal{R}(a, b)$, we can derive useful results. First, it is apparent that once A_0 and A_1 are chosen, all remaining terms are completely determined by (1). This shows that $\mathcal{R}(a, b)$ is a two-dimensional space. Indeed, there is a natural basis $\{E, F\}$ where E has starting values 1 and 0, and F, with starting values 0 and 1, is the (a, b)-Fibonacci sequence. Thus

$$E = 1, 0, b, ab, a^2b + b^2, \ldots$$
$$F = 0, 1, a, a^2 + b, a^3 + 2ab, \ldots.$$

Clearly, $A = A_0E + A_1F$ for all $A \in \mathcal{R}(a, b)$. Note further that $\Lambda E = bF$, so that we can easily express any A just using F:

$$A_n = bA_0F_{n-1} + A_1F_n \tag{4}$$

As an element of $\mathcal{R}(a, b)$, L can thus be expressed in terms of F. From (4), we have

$$L_n = 2bF_{n-1} + aF_n.$$

But the fundamental recursion (1) then leads to

$$L_n = bF_{n-1} + F_{n+1}. \tag{5}$$

This is the analog of the Lucas-Fibonacci connection stated above.

Recall that the difference and the k-term sum operators all preserve $\mathcal{R}(a, b)$. Thus, ΔF and $\Sigma_k F$ are elements of $\mathcal{R}(a, b)$ and can be expressed in terms of F using (4). The case for Σ is a more interesting application of operator methods. The question is this: If $A \in \mathcal{R}(a, b)$, what can we say about ΣA?

As a preliminary step, notice that a sequence is constant if and only if it is annihilated by the difference operator Δ. Now, suppose that $A \in \mathcal{R}(a, b)$. That means $(\Lambda^2 - a\Lambda - b)A = 0$, and so too

$$\Lambda(\Lambda^2 - a\Lambda - b)A = 0.$$

Now commute Λ with the other operator, and use (3) to obtain

$$(\Lambda^2 - a\Lambda - b)\Delta\Sigma A = 0.$$

Finally, since Δ and Λ commute, pull Δ all the way to the front to obtain

$$\Delta(\Lambda^2 - a\Lambda - b)\Sigma A = 0.$$

This shows that while $(\Lambda^2 - a\Lambda - b)\Sigma A$ may not be 0 (indicating $\Sigma A \notin \mathcal{R}(a, b)$), at worst it is constant. Now it turns out that there are two cases. If $a + b \neq 1$, it can be shown that ΣA differs from an element of $\mathcal{R}(a, b)$ by a constant. That tells us at once that there is an identity of the form

$$(\Sigma A)_n = c_0 F_n + c_1 F_{n-1} + c_2,$$

which corresponds to the running sum property for Fibonacci numbers. We will defer the determination of the constants c_i to the section on Binet formulas.

Here is the verification that ΣA differs from an element of $\mathcal{R}(a, b)$ by a constant when $a + b \neq 1$. We know that $(\Lambda^2 - a\Lambda - b)\Sigma A$ is a constant c_1. Suppose that we can find another constant, c, such that $(\Lambda^2 - a\Lambda - b)c = c_1$. Then we would have $(\Lambda^2 - a\Lambda - b)(\Sigma A - c) = 0$, hence $\Sigma A - c \in \mathcal{R}(a, b)$. It is an exercise to show c can be found exactly when $a + b \neq 1$.

When $a + b = 1$ we have the second case. A little experimentation with (1) will show you that in this case $\mathcal{R}(a, b)$ includes all the constant sequences. The best way to analyze this situation is to develop some general methods for solving difference equations. We do that next.

Binet formulas We mentioned earlier that there is a perfect analogy between linear constant coefficient difference and differential equations. In the differential equation case, a special role is played by the exponential functions, $e^{\lambda t}$, which are eigenvectors for the differential operator: $De^{\lambda t} = \lambda \cdot e^{\lambda t}$. For difference equations, the analogous role is played by the geometric progressions, $A_n = \lambda^n$. These are eigenvectors for the left-shift operator: $\Lambda\lambda^n = \lambda \cdot \lambda^n$. Both differential and difference equations can be formulated in terms of polynomials in the fundamental operator, Λ or D, respectively. These are in fact characteristic polynomials—the roots λ are eigenvalues and correspond to eigenvector solutions to the differential or difference equation. Moreover, except in the case of repeated roots, this leads to a basis for the space of all solutions.

We can see how this all works in detail in the case of $\mathcal{R}(a, b)$, which is viewed as the null space of $p(\Lambda) = \Lambda^2 - a\Lambda - b$. When is the geometric progression $A_n = \lambda^n$ in this null space? We demand that $A_{n+2} - aA_{n+1} - bA_n = 0$, so the condition is

$$\lambda^{n+2} - a\lambda^{n+1} - b\lambda^n = 0.$$

Excluding the case $\lambda = 0$, which produces only the trivial solution, this leads to $p(\lambda) = 0$ as a necessary and sufficient condition for $\lambda^n \in \mathcal{R}(a, b)$. Note also that the roots of p are related to the coefficients in the usual way. Thus, if the roots are λ and μ, then

$$\lambda + \mu = a \tag{6}$$

$$\lambda\mu = -b. \tag{7}$$

Now if λ and μ are distinct, then λ^n and μ^n are independent solutions to the difference equation. And since we already know that the null space is two dimensional, that makes $\{\lambda^n, \mu^n\}$ a basis. In this case, $\mathcal{R}(a, b)$ is characterized as the set of linear combinations of these two geometric progressions. In particular, for $A \in \mathcal{R}(a, b)$, we can express A in the form

$$A_n = c_\lambda \lambda^n + c_\mu \mu^n. \tag{8}$$

The constants c_λ and c_μ are determined by the initial conditions

$$A_0 = c_\lambda + c_\mu$$

$$A_1 = c_\lambda \lambda + c_\mu \mu.$$

We are assuming λ and μ are distinct, so this system has the solution

$$c_\lambda = \frac{A_1 - \mu A_0}{\lambda - \mu}$$

$$c_\mu = \frac{\lambda A_0 - A_1}{\lambda - \mu}.$$

Now let us apply these to the special cases of F and L. For F, the initial values are 0 and 1, so $c_\lambda = 1/(\lambda - \mu)$ and $c_\mu = -1/(\lambda - \mu)$. For L the initial terms are 2 and $a = \lambda + \mu$. This gives $c_\lambda = c_\mu = 1$. Thus,

$$F_n = \frac{\lambda^n - \mu^n}{\lambda - \mu} \tag{9}$$

$$L_n = \lambda^n + \mu^n. \tag{10}$$

These are the *Binet Formulas* for $\mathcal{R}(a, b)$.

When $\lambda = \mu$, the fundamental solutions of the difference equation are $A_n = \lambda^n$ and $B_n = n\lambda^n$. Most of the results for $\mathcal{R}(a, b)$ have natural extensions to this case. For example, in the case of repeated root λ, the Binet formulas become

$$F_n = n\lambda^{n-1}$$

$$L_n = 2\lambda^n.$$

Extensions of this sort are generally quite tractable, and we will not typically go into the details. Accordingly, we will assume from now on that p has distinct roots, or equivalently, that $a^2 + 4b \neq 0$.

Another special case of interest occurs when one root is 1. In this case, the geometric progression 1^n is constant, and $\mathcal{R}(a, b)$ contains all the constant sequences. As we saw earlier, the condition for this is $a + b = 1$. Now the Binet representation gives a new way of thinking about this result. It is an exercise to verify that $a + b = 1$ if and only if 1 is a root of p.

If both roots equal 1, the fundamental solutions are $A_n = 1$ and $B_n = n$. This shows that $\mathcal{R}(2, -1)$ consists of all the arithmetic progressions, confirming our earlier observation for Example 2.

Let us revisit the other examples presented earlier, and consider the Binet formulas for each.

Example 0: $\mathcal{R}(1, 1)$. For the normal Fibonacci and Lucas numbers, $p(t) = t^2 - t - 1$, and the roots are α and β as defined earlier. The general version of the Binet formulas reduce to the familiar form upon substitution of α and β for λ and μ.

Example 1: $\mathcal{R}(11, -10)$. Here, with $p(t) = t^2 - 11t + 10$, the roots are 10 and 1. In this case the Binet formulas simply tell us what is already apparent: $F_n = (10^n - 1)/9$ and $L_n = 10^n + 1$.

Example 3: $\mathcal{R}(3, -2)$. In this example, $p(t) = t^2 - 3t + 2$, with roots 2 and 1. The Binet formulas confirm the pattern we saw earlier: $F_n = 2^n - 1$ and $L_n = 2^n + 1$.

Example 4: $\mathcal{R}(1, -1)$. Now $p(t) = t^2 - t + 1$. Note that $p(t)(t + 1) = t^3 + 1$, so that roots of p are cube roots of -1 and hence, sixth roots of 1. This explains the periodic nature of F and L. Indeed, since $\lambda^6 = \mu^6 = 1$ in this case, every element of $\mathcal{R}(1, -1)$ has period 6 as well.

Example 5: $\mathcal{R}(3, -1)$. The roots in this example are α^2 and β^2. The Binet formulas involve only even powers of α and β, hence the appearance of the even Fibonacci and Lucas numbers.

Example 6: $\mathcal{R}(4, 1)$. This example is similar to the previous one, except that the roots are α^3 and β^3.

Example 7: $\mathcal{R}(2, 1)$. For this example $p(t) = t^2 - 2t - 1$, so the roots are $1 \pm \sqrt{2}$. The Binet formulas give

$$F_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}} \quad \text{and} \quad L_n = (1+\sqrt{2})^n + (1-\sqrt{2})^n.$$

Characterizing $\mathcal{R}(a, b)$ in terms of geometric progressions has immediate applications. For example, consider the ratio of successive terms of a sequence in $\mathcal{R}(a, b)$. Using (8), we have

$$\frac{A_{n+1}}{A_n} = \frac{c_\lambda \lambda^{n+1} + c_\mu \mu^{n+1}}{c_\lambda \lambda^n + c_\mu \mu^n}.$$

Now assume that $|\lambda| > |\mu|$, and divide out λ^n:

$$\frac{A_{n+1}}{A_n} = \frac{c_\lambda \lambda + c_\mu \mu (\mu^n/\lambda^n)}{c_\lambda + c_\mu (\mu^n/\lambda^n)}.$$

Since $(\mu/\lambda)^n$ will go to 0 as n goes to infinity, we conclude

$$\frac{A_{n+1}}{A_n} \to \lambda \quad \text{as} \quad n \to \infty.$$

In words, the ratio of successive terms of a sequence in $\mathcal{R}(a, b)$ always tends toward the dominant eigenvalue as n goes to infinity. That is the general version of the asymptotic behavior we observed for Fibonacci numbers.

As a second example, if $A_n = c_\lambda \lambda^n + c_\mu \mu^n$, then $\Omega_k A_n = c_\lambda \lambda^{kn} + c_\mu \mu^{kn}$. This is a linear combination of two geometric progressions as well, with eigenvalues λ^k and μ^k. Consequently, $\Omega_k A \in \mathcal{R}(a', b')$ for some a' and b'. Now using the relationship between roots and coefficients again, we deduce that $a' = \lambda^k + \mu^k$, and by (10) that gives $a' = L_k^{(a,b)}$. Similarly, we find $b' = -(\lambda\mu)^k = -(-b)^k$. Thus,

$$\Omega_k : \mathcal{R}(a, b) \to \mathcal{R}(L_k^{(a,b)}, -(-b)^k). \tag{11}$$

We can extend this slightly. If $A \in \mathcal{R}(a, b)$, then so is $\Lambda^d A$ for any positive integer d. Thus, $\Omega_k \Lambda^d A \in \mathcal{R}(a', b')$. In other words, when $A \in \mathcal{R}(a, b)$, the sequence $B_n = A_{kn+d}$ is in $\mathcal{R}(a', b')$. This corresponds to sampling A at the terms of an arithmetic progression.

In the particular case of F and L, we can use the preceding results to determine the effect of Ω_k explicitly. For notational simplicity, we will again denote by a' and b' the values $L_k^{(a,b)}$ and $-(-b)^k$, respectively. We know that $\Omega_k F^{(a,b)} \in \mathcal{R}(a', b')$, and begins with the terms 0 and $F_k^{(a,b)}$. This is necessarily a multiple of $F^{(a',b')}$, and in particular, gives

$$\Omega_k F^{(a,b)} = F_k^{(a,b)} \cdot F^{(a',b')}. \tag{12}$$

Similarly, $\Omega_k L^{(a,b)}$ begins with 2 and $L_k^{(a,b)}$. But remember that the latter of these is exactly $a' = L_1^{(a',b')}$. Thus,

$$\Omega_k L^{(a,b)} = L^{(a',b')}. \tag{13}$$

Of course, this last equation is easily deduced directly from the Binet formula for $L^{(a,b)}$, as well. The observations in Examples 5 and 6 are easily verified using (12) and (13).

For one more example, let us return to the analysis of ΣA for $A \in \mathcal{R}(a, b)$. Again using the expression $A_n = c_\lambda \lambda^n + c_\mu \mu^n$ we find the terms of ΣA as

$$\Sigma A_n = c_\lambda \frac{\lambda^{n+1} - 1}{\lambda - 1} + c_\mu \frac{\mu^{n+1} - 1}{\mu - 1}.$$

Evidently, this is invalid if either λ or μ equals 1. So, as before, we exclude that possibility by assuming $a + b \neq 1$.

Under this assumption, we found earlier that ΣA must differ from an element of $\mathcal{R}(a, b)$ by a constant. Now we can easily determine the value of that constant. Rearranging the preceding equation produces

$$\Sigma A_n = \frac{c_\lambda \lambda}{\lambda - 1} \lambda^n + \frac{c_\mu \mu}{\mu - 1} \mu^n - \left(\frac{c_\lambda}{\lambda - 1} + \frac{c_\mu}{\mu - 1} \right).$$

This clearly reveals ΣA as the sum of an element of $\mathcal{R}(a, b)$ with the constant $C = -(c_\lambda/(\lambda - 1) + c_\mu/(\mu - 1))$.

In general, the use of this formula requires expressing A in terms of λ and μ. But in the special case of F, we can express the formula in terms of a and b. Recall that when $A = F$, $c_\lambda = 1/(\lambda - \mu)$ and $c_\mu = -1/(\lambda - \mu)$. Substituting these in the earlier formula for C, leads to

$$\begin{aligned} C &= -\frac{1}{\lambda - \mu} \left(\frac{1}{\lambda - 1} - \frac{1}{\mu - 1} \right) \\ &= -\frac{1}{\lambda - \mu} \frac{\mu - \lambda}{(\lambda - 1)(\mu - 1)} = \frac{1}{(\lambda\mu - \lambda - \mu + 1)}. \end{aligned}$$

Once again using (6) and (7), this yields

$$C = \frac{1}{1 - a - b}. \tag{14}$$

As an example, let us consider ΣF for $\mathcal{R}(2, 3)$. In the table below, the first several terms of F and ΣF are listed.

n	0	1	2	3	4	5
F_n	0	1	2	7	20	61
ΣF_n	0	1	3	10	30	91

In this example, we have $C = 1/(1 - 2 - 3) = -1/4$. Accordingly, adding 1/4 to each term of ΣF should produce an element of $\mathcal{R}(2, 3)$. Carrying out this step produces

$$\Sigma F + \frac{1}{4} = \frac{1}{4}(1, 5, 13, 41, 121, 365, \ldots).$$

As expected, this is an element of $\mathcal{R}(2, 3)$.

Applying a similar analysis in the general case (with the assumption $a + b \neq 1$) leads to the identity

$$\Sigma F_n = \frac{1}{a+b-1}(F_{n+1} + bF_n - 1).$$

This reduces to the running sum property for Fibonacci numbers when $a = b = 1$. A similar analysis applies in the case $a + b = 1$. We leave the details to the reader.

In the derivation of the Binet formulas above, a key role was played by the eigenvectors and eigenvalues of the shift operator. It is therefore not surprising that there is a natural matrix formulation of these ideas. That topic is the third general context for tool development.

Matrix formulation Using the natural basis $\{E, F\}$ for $\mathcal{R}(a, b)$, we can represent Λ by a matrix M. We already have seen that $\Lambda E = bF$, so the first column of M has entries 0 and b. Applying the shift to F produces $(1, a, \ldots) = E + aF$. This identifies the second column entries of M as 1 and a, so

$$M = \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}. \tag{15}$$

Now if $A \in \mathcal{R}(a, b)$, then relative to the natural basis it is represented by $[A] = [A_0 \;\; A_1]^T$. Similarly, the basis representation of $\Lambda^n A$ is $[A_n \;\; A_{n+1}]^T$. On the other hand, we can find the same result by applying M n times to $[A]$. Thus, we obtain

$$\begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}^n \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = \begin{bmatrix} A_n \\ A_{n+1} \end{bmatrix}. \tag{16}$$

Premultiplying by $[1 \;\; 0]$ then gives

$$\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}^n \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = A_n. \tag{17}$$

This gives a matrix representation for A_n.

Note that in general, the ith column of a matrix M can be expressed as the product $M\mathbf{e}_i$, where $\mathbf{e}_i$ is the ith standard basis element. But here, the standard basis elements are representations $[E]$ and $[F]$. In particular, $M^n[E] = [E_n \;\; E_{n+1}]^T$ and $M^n[F] = [F_n \;\; F_{n+1}]^T$. This gives us the columns of M^n, and therefore

$$M^n = \begin{bmatrix} E_n & F_n \\ E_{n+1} & F_{n+1} \end{bmatrix}.$$

Then, using $\Lambda E = bF$, we have

$$\begin{bmatrix} 0 & 1 \\ b & a \end{bmatrix}^n = \begin{bmatrix} bF_{n-1} & F_n \\ bF_n & F_{n+1} \end{bmatrix}. \tag{18}$$

This is the general version of the matrix form for Fibonacci numbers.

The matrix form leads immediately to two other properties. First, taking the determinant of both sides of (18), we obtain

$$bF_{n-1}F_{n+1} - bF_n^2 = (-b)^n.$$

Simplifying,

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n b^{n-1},$$

the general version of Cassini's formula.

Second, start with $M^n = M^m M^{n-m}$, expressed explicitly in the form

$$\begin{bmatrix} bF_{n-1} & F_n \\ bF_n & F_{n+1} \end{bmatrix} = \begin{bmatrix} bF_{m-1} & F_m \\ bF_m & F_{m+1} \end{bmatrix} \begin{bmatrix} bF_{n-m-1} & F_{n-m} \\ bF_{n-m} & F_{n-m+1} \end{bmatrix}.$$

By inspection, we read off the 1, 2 entry of both sides, obtaining

$$F_n = F_m F_{n-m+1} + bF_{m-1}F_{n-m}, \tag{19}$$

generalizing the *Convolution Property* for regular Fibonacci numbers. As a special case, replace n with $2n+1$ and m with $n+1$, producing

$$F_{2n+1} = F_{n+1}^2 + bF_n^2. \tag{20}$$

This equation will be applied in the discussion of Pythagorean triples.

This concludes our development of general tools. Along the way, we have found natural extensions of all but two of our famous Fibonacci properties. These extensions are all simple and direct consequences of the basic ideas in three general contexts: difference operators, Binet formulas, and matrix methods. Establishing analogs for the remaining two properties is just a bit more involved, and we focus on them in the next section.

The last two properties

Pythagorean triples In a way, the connection with Pythagorean triples is trivial. The well-known parameterization $(x^2 - y^2, 2xy, x^2 + y^2)$ expresses primitive Pythagorean triples in terms of quadratic polynomials in two variables. The construction using Fibonacci numbers is similar. To make this clearer, note that if w, x, y, z are four consecutive Fibonacci numbers, then we may replace w with $y - x$ and z with $y + x$. With these substitutions, the Fibonacci parameterization given earlier for Pythagorean triples becomes

$$(wz, 2xy, yz - wx) = \big((y-x)(y+x), 2yx, y(y+x) - x(y-x)\big).$$

Since we can reduce the parameterization to a quadratic combination of two parameters in this way, the ability to express Pythagorean triples loses something of its mystery. In fact, if w, x, y, z are four consecutive terms of any sequence in $\mathcal{R}(a, b)$, we may regard x and y as essentially arbitrary, and so use them to define a Pythagorean triple $(x^2 - y^2, 2xy, x^2 + y^2)$. Thus, we can construct a Pythagorean triple using *just two* consecutive terms of a Fibonacci-like sequence.

Is that cheating? It depends on what combinations of the sequence elements are considered legitimate. The Fibonacci numbers have been used to parameterize Pythagorean triples in a variety of forms. The version given above, $(wz, 2xy, yz - wx)$, appears in Koshy [**16**] with a 1968 attribution to Umansky and Tallman. Here and below we use consecutive letters of the alphabet rather than the original subscript formulation, as a notational convenience. Much earlier, Raine [**19**] gave it this way: $(wz, 2xy, t)$, where, if w is F_n then t is F_{2n+3}. Boulger [**1**] extended Raine's results and observed that the triple can also be expressed $(wz, 2xy, x^2 + y^2)$. Horadam [**13**] reported it in the form $(xw, 2yz, 2yz + x^2)$. These combinations use a variety of different quadratic monomials, including both yz and x^2. So, if those are permitted, why not simply use the classical $(x^2 - y^2, 2xy, x^2 + y^2)$ and be done with it? The more complicated parameterizations we have cited then seem to be merely exercises in complexification.

In light of these remarks, it should be no surprise that the Fibonacci parameterization of Pythagorean triples can be generalized to $\mathcal{R}(a, b)$. For example, Shannon and Horadam [**20**] give the following version: $((a/b^2)xw, 2Pz(Pz - x), x^2 + 2Pz(Pz - x))$ where $P = (a^2 - -b)/2b^2$.

Using a modified version of the diophantine equation, we can get closer to the simplicity of Raine's formulation. For $\mathcal{R}(a, b)$ we replace the Pythagorean identity with

$$X^2 + bY^2 = Z^2 \tag{21}$$

and observe that the parameterization

$$(X, Y, Z) = (v^2 - bu^2, 2uv, v^2 + bu^2)$$

always produces solutions to (21). Now, if w, x, y, and z are four consecutive terms of $A \in \mathcal{R}(a, b)$, then we can express the first and last as

$$w = \frac{1}{b}(y - ax)$$
$$z = bx + ay.$$

Define constants $c = b/a$ and $d = c - a$. Then a calculation verifies that

$$(X, Y, Z) = (cwz - dxy, 2xy, xz + bwy) \tag{22}$$

is a solution to (21). In fact, with $u = x$ and $v = y$, it is exactly the parameterization given above.

In the special case that $x = F_n^{(a,b)}$, we can also express (22) in the form

$$(X, Y, Z) = (cwz - dxy, 2xy, t)$$

where $t = F_{2n+1}$. This version, which generalizes the Raine result, follows from (20). Note, also, that when $a = b = 1$, (22) becomes $(wz, 2xy, xz + wy)$, which is another variant on the Fibonacci parameterization of Pythagorean triples.

Greatest common divisor The Fibonacci properties considered so far make sense for real sequences in $\mathcal{R}(a, b)$. Now, however, we will consider divisibility properties that apply to integer sequences. Accordingly, we henceforth assume that a and b are integers, and restrict our attention to sequences $A \in \mathcal{R}(a, b)$ for which the initial terms A_0 and A_1 are integers, as well. Evidently, this implies A is an integer sequence. In order to generalize the gcd property, we must make one additional assumption: that a and b are relatively prime. Then we can prove in $\mathcal{R}(a, b)$, that the gcd of F_m and F_n is F_k, where k is the gcd of m and n. The proof has two parts: We show that F_k is a divisor of both F_m and F_n, and that F_m/F_k and F_n/F_k are relatively prime. The first of these follows immediately from an observation about the skip operator already presented. The second part depends on several additional observations.

OBSERVATION 1. *F_k is a divisor of F_{nk} for all $n > 0$.*

Proof. We have already noted that $\Omega_k F = F_k \cdot F^{(a',b')}$ so every element of $\Omega_k F = F_0, F_k, F_{2k}, \dots,$ is divisible by F_k.

OBSERVATION 2. *F_n and b are relatively prime for all $n \geq 0$.*

Proof. Suppose p is prime divisor of b. Since a and b are relatively prime, p is not a divisor of a. Modulo p, the fundamental recursion (1) becomes $F_{n+2} \equiv aF_{n+1}$, so $F_n \equiv F_1 a^{n-1}$ for $n \geq 1$. This shows that $F_n \not\equiv 0$, since p is not a divisor of a.

OBSERVATION 3. *If $A \in \mathcal{R}(a, b)$, and if p is a common prime divisor of A_k and A_{k+1}, but is not a divisor of b, then p is a divisor of A_n for all $n \geq 0$.*

Proof. If $k > 0$, $A_{k+1} = aA_k + bA_{k-1}$, so p is a divisor of A_{k-1}. By induction, p divides both A_0 and A_1, and therefore A_n for all $n \geq 0$.

OBSERVATION 4. *If positive integers h and k are relatively prime, then so are F_h and F_k.*

Proof. If p is a prime divisor of F_h and F_k, then by Observation 2, p is not a divisor of b. Since h and k are relatively prime, there exist integers r and s such that $rh + sk = 1$. Clearly r and s must differ in sign. Without loss of generality, we assume that $r < 0$, and define $t = -r$. Thus, $sk - th = 1$. Now by Observation 1, F_{sk} is divisible by F_k, and hence by p. Similarly, F_{th} is divisible by F_h, and hence, also by p. But F_{th} and F_{sk} are consecutive terms of F, so by Observation 3, p is a divisor of all F_n. That is a contradiction, and shows that F_h and F_k can have no common prime divisor.

OBSERVATION 5. *If $a' = L_k^{(a,b)}$ and $b' = -(-b)^k$, then a' and b' are relatively prime.*

Proof. Suppose, to the contrary, that p is a common prime divisor of a' and b'. Then clearly p is a divisor of b, and also a divisor of $L_k^{(a,b)}$, which equals $bF_{k-1}^{(a,b)} + F_{k+1}^{(a,b)}$ by (5). This makes p a divisor of $F_{k+1}^{(a,b)}$, which contradicts Observation 2.

With these observations, we now can prove the

THEOREM. *The* gcd *of F_m and F_n is F_k, where k is the* gcd *of m and n.*

Proof. Let $s = m/k$ and $t = n/k$, and observe that s and t are relatively prime. We consider $A = \Omega_k F = F_0, F_k, F_{2k}, \ldots$. As discussed earlier, A can also be expressed as $F_k \cdot F^{(a',b')}$ where $a' = L_k$ and $b' = -(-b)^k$. Moreover, by Observation 5, a' and b' are relatively prime. As in Observation 1, we see at once that every A_j is a multiple of F_k, so in particular, F_k is a divisor of $A_s = F_{ks} = F_m$ and $A_t = F_{kt} = F_n$. On the other hand, $F_m/F_k = F_s^{(a',b')}$ and $F_n/F_k = F_t^{(a',b')}$, are relatively prime by Observation 4. Thus, F_k is the gcd of F_m and F_n. ■

Several remarks about this result are in order. First, in Michael [**18**], the corresponding result is established for the traditional Fibonacci numbers. That proof depends on the $\mathcal{R}(1, 1)$ instances of (19), Observation 1, and Observation 3, and extends to a proof for $\mathcal{R}(a, b)$ in a natural way.

Second, Holzsager [**10**] has described an easy construction of other sequences A_n for which $\gcd(A_n, A_m) = A_{\gcd(m,n)}$. First, for the primes p_k, define $A_{p_k} = q_k$ where the q_k are relatively prime. Then, extend A to the rest of the integers multiplicatively. That is, if $n = \prod p_i^{e_i}$ then $A_n = \prod q_i^{e_i}$. Such a sequence defines a mapping on the positive integers that carries the prime factorization of any subscript into a corresponding factorization involving the qs. This mapping apparently will commute with the gcd. By Observation 4, the terms $F_{p_k}^{(a,b)}$ are relatively prime, but since $F_2 = a$ and $F_4 = a^3 + 2ab$, the F mapping is not generally multiplicative. Thus, Holzsager's construction does not lead to examples of the form $F^{(a,b)}$.

Finally, we note that there is similar result for the (a, b)-Lucas numbers, which we omit in the interest of brevity. Both that result and the preceding Theorem also appear in Hilton and Pedersen [**8**]. Also, the general gcd result for F was known to Lucas, and we may conjecture that he knew the result for L as well.

In particular and in general

We have tried to show in this paper that much of the mystique of the Fibonacci numbers is misplaced. Rather than viewing F as a unique sequence with an amazing host of algebraic, combinatorial, and number theoretic properties, we ought to recognize that it is simply one example of a large class of sequences with such properties. In so arguing, we have implicitly highlighted the tension within mathematics between the particular and the general. Both have their attractions and pitfalls. On the one hand, by focusing too narrowly on a specific amazing example, we may lose sight of more general principles at work. But there is a countervailing risk that generalization may add nothing new to our understanding, and result in meaningless abstraction.

In the case at hand, the role of the skip operator should be emphasized. The proof of the gcd result, in particular, was simplified by the observation that the skip maps one $\mathcal{R}(a, b)$ to another. This observation offers a new, simple insight about the terms of Fibonacci sequences—an insight impossible to formulate without adopting the general framework of two-term recurrences.

It is not our goal here to malign the Fibonacci numbers. They constitute a fascinating example, rich with opportunities for discovery and exploration. But how much more fascinating it is that an entire world of such sequences exists. This world of the super sequences should not be overlooked.

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