The Pythagonacci Family Reunion

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The Amazing Connection

- w, x, y, and z are any four consecutive Fibonacci numbers
- a = product of outers (wz)
- b = twice inners $(2xy)$
- c = odds plus evens $(wy + xz)$
- (a, b, c) is a Pythagorean Triple!
- O2IOPE!

OTIFAL

- w, x, y, and z are any four consecutive Fibonacci numbers
- $a =$ product of Outers (wz)
- $b =$ Twice Inners ($2xy$)
- $c =$ Firsts And Lasts ($wy + xz$)
- (a, b, c) is a Pythagorean Triple!
- OTIFAL!

Examples

- $(w, x, y, z) = (1, 2, 3, 5)$
- $a = 5$ $\quad$ $(outers)$
- $b = 12$ $\quad$ $(twice\ inners)$
- $c = 13$ $\quad$ $(firsts\ and\ lasts)$

- $(w, x, y, z) = (3, 5, 8, 13)$
- $a = 39$
- $b = 80$
- $c = 89$

History

- $(wz, 2xy, x^2 + y^2)$ Boulger, *Pythagoras Meets Fibonacci*, 1989
- $(F_n F_{n+3}, 2F_{n+1}F_{n+2}, F_{2n+3})$ Charles Raine, 1948
- $(wz, 2xy, 2xy + w^2)$, Horadam, 1961
- $(wz, 2xy, yz - wx)$, Umansky & Tallman, 1968
- $(wz, 2xy, wy + xz)$, Mena & Kalman, 2003 (O2IOPE!)

Proving O2IOPE

- $(w, x, y, z) = (y - x, x, y, x + y)$
- Outers: $y^2 - x^2$
- Twice Inners: $2xy$
- Odds + Evens: $y^2 - xy + x^2 + xy = y^2 + x^2$
- PT: $(y^2 - x^2, 2xy, y^2 + x^2)$

Generalizing O2IOPE

- Is this *amazing connection* a peculiarity of the Fibonacci numbers?
- Proof shows property holds for any sequence $\{t_n\}$ satisfying $t_{n+2} = t_{n+1} + t_n$
- How about creating Pythagorean Triples from *three* consecutive Fibonacci numbers? $(x, y, z) = (x, y, x + y)$
- How about just TWO consecutive Fibonacci numbers?
- How about just ONE?!?!?!?

ab-Fibonacci Numbers

- General linear second order recursions: $t_{n+2} = at_{n+1} + bt_n$
- Class of all such sequences $\mathcal{R}(a, b)$ in Kalman-Mena paper
- $ab-$Fibonacci numbers: $F_n \in \mathcal{R}(a, b)$ given by $0, 1, a, a^2 + b, a^3 + 2ab, \cdots$
- Shannon and Horadam's $ab-$Fibonacci Pythagorean connection
- A new connection

Shannon and Horadam

- (w, x, y, z) consecutive terms of a sequence $\{t_n\} \in \mathcal{R}(a, b)$
- $\xi = a/b^2$
- $\eta = (a^2 + b)/2b^2$
- $(\xi wz,\ \ 2\eta y(\eta y - w),\ \ w^2 + 2\eta y(\eta y - w))$ is a PT

My Own Version

- $b = k^2 + 2ak$ for arbitrary natural number k
- $\xi = b$
- $\eta = a^2 + b$
- $\theta = 2ab(a + k)$
- $\zeta = 2a^2b + b^2$
- $\eta = (a^2 + b)/2b^2$
- $(\xi wz - \eta xy,\ \ \theta wx,\ \ yz - \zeta wx)$ is a PT

Meet the Pythagorean Family

- Generalized Equation: $X^2 + bY^2 = Z^2$
- Parameterized Solution:$(y^2 - bx^2, 2xy, y^2 + bx^2)$
- Call this a b-Pythagorean triple
- We can construct b-Pythagorean triples using ab-Fibonacci numbers

b-Pythagoreans and ab-Fibonaccis

- Let w, x, y, z be 4 consecutive terms of $\{t_n\} \in \mathcal{R}(a, b)$
- Define $\xi = b/a$ and $\eta = \xi - a$
- $(\xi wz - \eta xy, 2xy, xz + bwy)$ is a b-PT

- Special case: $a = 1$
- $(bwz - (b - a)xy, 2xy, xz + bwy)$ is a b-PT
- If $w = F_n$ then $xz + bwy = F_{2n+3}$ as in Raine result
- $a = b = 1 : (wz, 2xy, xz + wy)$ is a PT

Example: $\mathcal{R}(1,3)$

1. Sample sequence: 0, 1, 1, 4, 7, 19, ...
2. $\xi = b = 3$; $\eta = b - a = 2$
3. b-PT: $(3wz - 2xy, 2xy, xz + 3wy)$
4. Take $(w, x, y, z) = (1, 1, 4, 7)$
5. b-PT: $(13, 8, 19)$
6. $13^2 + 3 \cdot 8^2 = 19^2$

Proof:

The four consecutive terms can be written

$$(w, x, y, z) = ((y - ax)/b, x, y, ay + bx)$$

Then

$$\begin{aligned} \frac{b}{a}wz - (\frac{b}{a} - a)xy &= y^2 - bx^2 \\ 2xy &= 2xy \\ xz + bwy &= y^2 + bx^2 \end{aligned}$$

And that is the *standard* parameterization of b-Pythagorean Triples