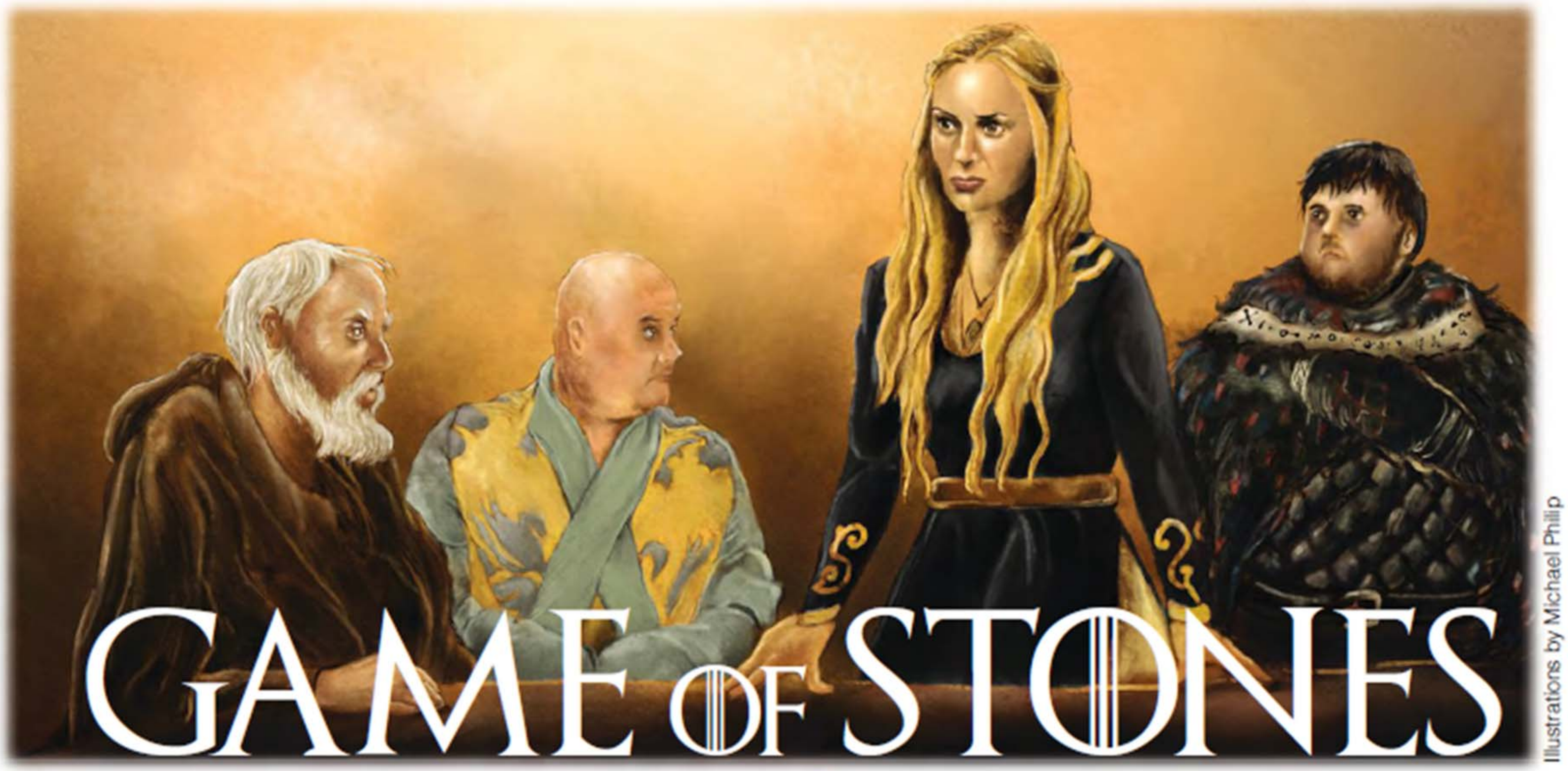


A Tour of Knights

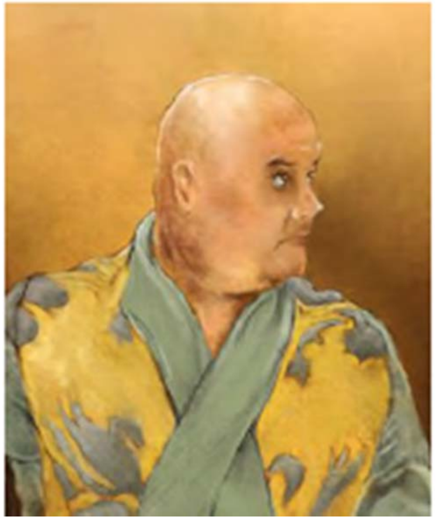
MH Article: joint with Dan Pritikin

This talk: joint with John Nolan
and Donna Dietz

I want them all DEAD!

- Queen Regent Kursei λ-stir
- 100 knights in the dungeon
- In spite of her usually sunny disposition and friendly, inclusive, compassionate attitude, she insists they are all traitors and should be executed.
- “May the blade of Payne be rusty and dull”

Caution, Your Grace



- Ferrous, Master of Whisperers
- “You can’t just go around beheading knights of the realm!”

- Maester π-celle
- “He’s right. There are rules and forms that must be observed.”
- What we need is some sort of trial, say a game of chance.

The Game of Stones

- Sumwell Tarley speaks up
- 100 knights names inscribed on 100 coffers and 100 stones.
- Stones randomly placed in coffers.
- Each knight permitted to open 50 coffers
- Success: finding his (or her) own name
- If all 100 knights succeed, all will be pardoned; if even 1 knight fails, all will be executed.
- Apparent odds of survival: 1 in 2^{100}

Details, Details

Ferrous: Too transparently stacked against the knights. They must have at least the appearance of a fair chance to survive.

π-celle: True. So let the knights have seven trials, in honor of the seven gods.

Kursei: [Thinks: so instead of one chance in 2^{100} it will be 7 chances in 2^{100}. Fine. They are still toast. Heh heh heh.] So be it. Every day for a week, we will mix up the stones and distribute them in the coffers. If the knights are truly innocent, surely *one* of the gods will intervene to save them … .

Tarley's Secret

- Sumwell Tarley sent to dungeon to inform the prisoners of the fate that awaits them
- He tells Ser Davos *C*-Worthy, the union knight, of a secret strategy.
- If you follow it, the probability that all knights will find their names on any given day is nearly 1/3.
- Davos realizes: We will all die if we fail seven times in a row. The probability of that is $(2/3)^7$. Thus we will all survive with probability $1 – (2/3)^7$ which is about .94!

How Can That Be?

Tarley: I don't know why it works. But I have simulated the strategy many times, with quite consistent results.

Davos: And how did you discover the strategy?

Tarley: The game of stones trial, and the strategy, are described in an ancient scroll fragment I found in the archives of Castle Black. The fragment did not contain any explanation. I hunted for other fragments for weeks, in vain.

Davos: No matter. If you say it works, I will trust you. What other option is there? [Dramatic Pause] BUT WHAT IS THE STRATEGY?

The Secret Strategy Revealed

"Here is what you do, Ser Davos. Open first the coffer with your own name. If it contains your name you have won, and your turn ends. But if you find Ser Jexian's stone, open his coffer next. Again, if you find your name, you win. If not, say you find the stone of Ser Reppatishus [The Recursion Knight] …"

"---- Yes, I get it, I look in *his* coffer. And does this strategy somehow lead me to my own name?"

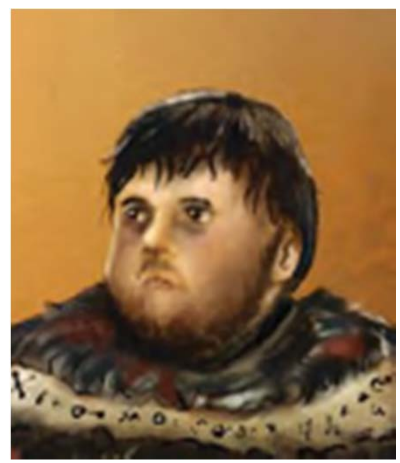
"Not necessarily. But you'll all find your own names about one time in three, based on my simulations."

"OK. We'll do it. I'll make sure everyone knows what to do, and by the seven, they WILL follow my orders."

A Midnight Visitor

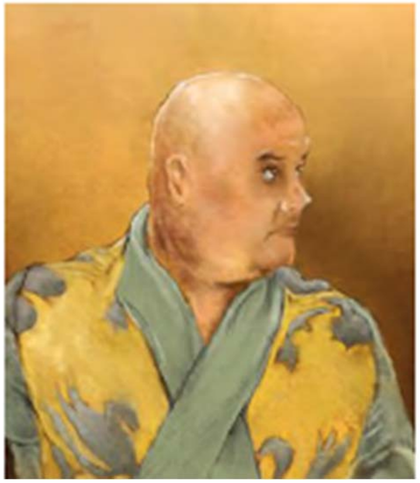
- A knock on the door, late at night. Who could it be?

- Sumwell Tarley cracks open his door. “Lord Ferrous! What brings you …”
- Ferrous steels in furtively. “Look here, Tarley. I want to know about the secret strategy.



- Tarley is near panic. “But, but, how do you, I mean, what are you talking about?”

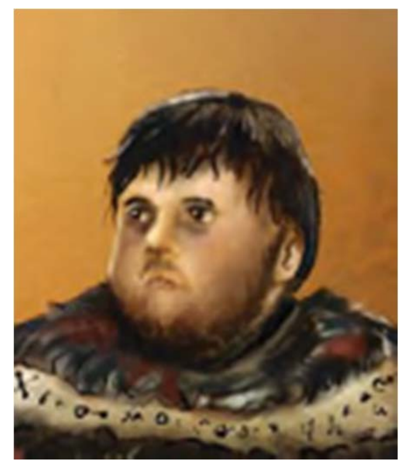
The Little Birds



Ferrous: Don't worry. I won't give you away. But I want to know about the strategy you revealed to Ser Davos.

Tarley: You knew?

Ferrous: Of course. My little birds go everywhere. But they don't always understand what they hear. Now what is this simulation you kept mentioning. Can you show me how it works?



The Simulation

- Tarley explained – you can play the game of stones with any even number of knights.
- For simplicity, we can simulate a game with 6 knights.
- Also, let us identify them with numbers, rather than names.
- We have 6 numbered coffers, and 6 numbered stones.
- To start the simulation, we randomly distribute the stones among the coffers.

Play out the game, starting with knight 1. But don’t stop after opening three coffers.

Knight 1 opens box 1 and finds a 6...

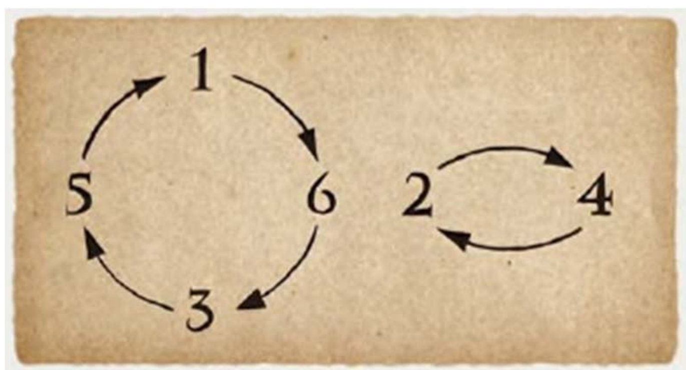
… opens box 6 and finds a 3 …

… opens box 3 and finds a 5 …

… opens box 5 and finds a 1.

- Knight 1: box 1 → box 6 → box 3 → box 5
- If he continues to follow the same process, he will endlessly follow the same cycle of boxes 1, 6, 3, 5, 1, 6, 3, 5, …
- Knights 6, 3, and 5 will follow the same cycle, but will each start with a different box.
- Each of these knights finds his own number after opening 4 boxes

- In a similar way, knights 2 and 4 will each follow the cycle 2, 4, 2, 4, 2, 4, …
- Each finds his own number after opening 2 boxes
- So we can see how the entire game plays out in this cycle diagram:



In General

- The ordering of the 6 stones in 6 coffers represents a permutation of {1, 2, 3, 4, 5, 6}
- For any permutation, following Sumwell's strategy must produce some diagram like the one we saw before. Ie, the permutation can be decomposed into a collection of disjoint cycles.
- The same thing holds for permutations of the set {1, 2, 3, … , 100} (or any other finite set).
- All 100 knights find their own stones after opening 50 boxes (or fewer) iff the cycles are all of length 50 or less.

Rusty Lord Ferrous

- Lord Ferrous concludes: the question is, if a permutation is chosen at random, how likely is it that the cycles are all of length 50 or less.
- “Let’s see… I fear I am a little rusty at combinatorial probability problems …
- Warm up with this: how likely is it that there will be a cycle of length 60?
- Notice: there can only be one such cycle, and the remaining cycles all have to be of length 40 or less.

Lord Ferrous Continues

- There are a total of 100! permutations. How many have a cycle of length 60?
- Can pick 60 cycle elements in $\binom{100}{60}$ ways.
- If we always start with the least one, there are 59! ways to order the remaining terms of the cycle.
- There are 40! different ways to permute the 40 elements not in the cycle.
- Conclusion: $\binom{100}{60}$ 59! 40! permutations have a cycle of length 60

Lord Ferrous Continues

- There are a total of 100! permutations.
- There are $\binom{100}{60}$ 59! 40! with a cycle of length 60.
- The probability that a random permutation will have a cycle of length 60 is

$$\binom{100}{60}\ 59!\,40!\ \div 100! = \frac{100!}{60!\,40!} \cdot \frac{59!\,40!}{100!}$$

- Massive cancellation!
- Prob = 1/60
- Likewise, a cycle of length *k* > 50 occurs with probability 1/*k*

Odds in Game of Stones

- The knights lose a round iff there is a cycle of length > 50
- Let p_k be the probability of a cycle of length k
- Prob(cycle of length > 50)

  $= p_{51} + p_{52} + p_{53} + \cdots + p_{100}$

  $= \frac{1}{51} + \frac{1}{52} + \frac{1}{53} + \cdots + \frac{1}{100}$
- Notice: this is approximately $\int_{50}^{100} \frac{1}{t} dt = \ln 2 \cong .69$
- Prob($\geq$1 win in 7 games) $\cong 1 - .69^7 \cong .93$

Max Cycle Length

- What if the knights can open more boxes? Fewer boxes?
- For any permutation s , we define mcl(s) = maximum cycle length of s.
- If knights can open k boxes, they will all succeed iff the stone-box permutation has mcl $\leq k$.
- We know prob(mcl = k) = $1/k$ if $51 \leq k \leq 100$
- What is the distribution for $k \leq 50$?

Partition Probabilities

- Consider a permutation of 100 objects with these cycle lengths: 3, 3, 3, 3, 3, 7, 7, 7, 12, 12, 40.
- This list of numbers is a *partition* of 100
- So: 5 3's, 3 7's, 2 12's, 1 40.
- Probability that a random permutation has this cycle structure is

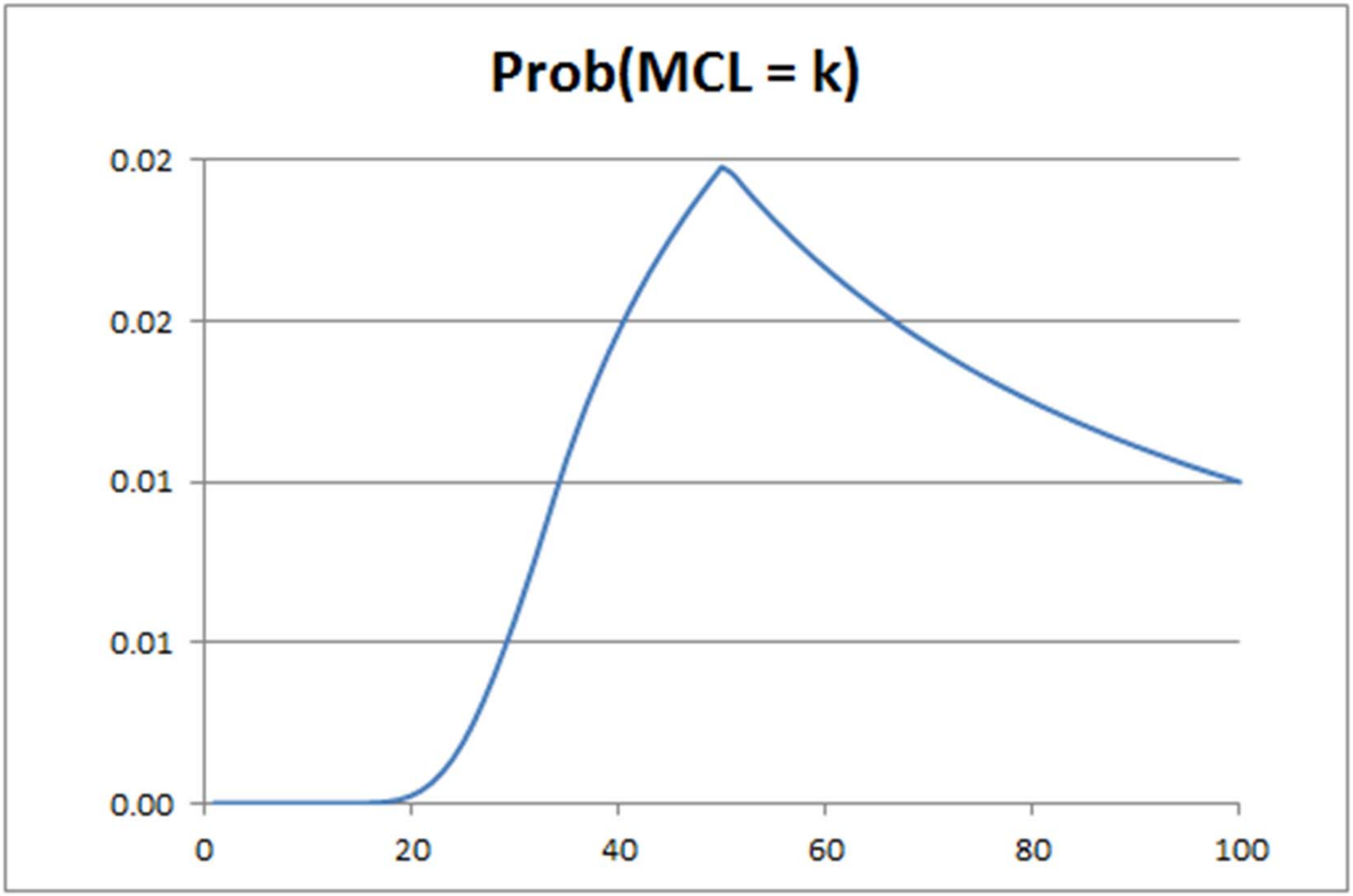
$$\frac{1}{3^5 7^3 12^2 40^1 \cdot 5!\,3!\,2!\,1!}$$

- Routine combinatorics computation

Probability that MCL = *k*

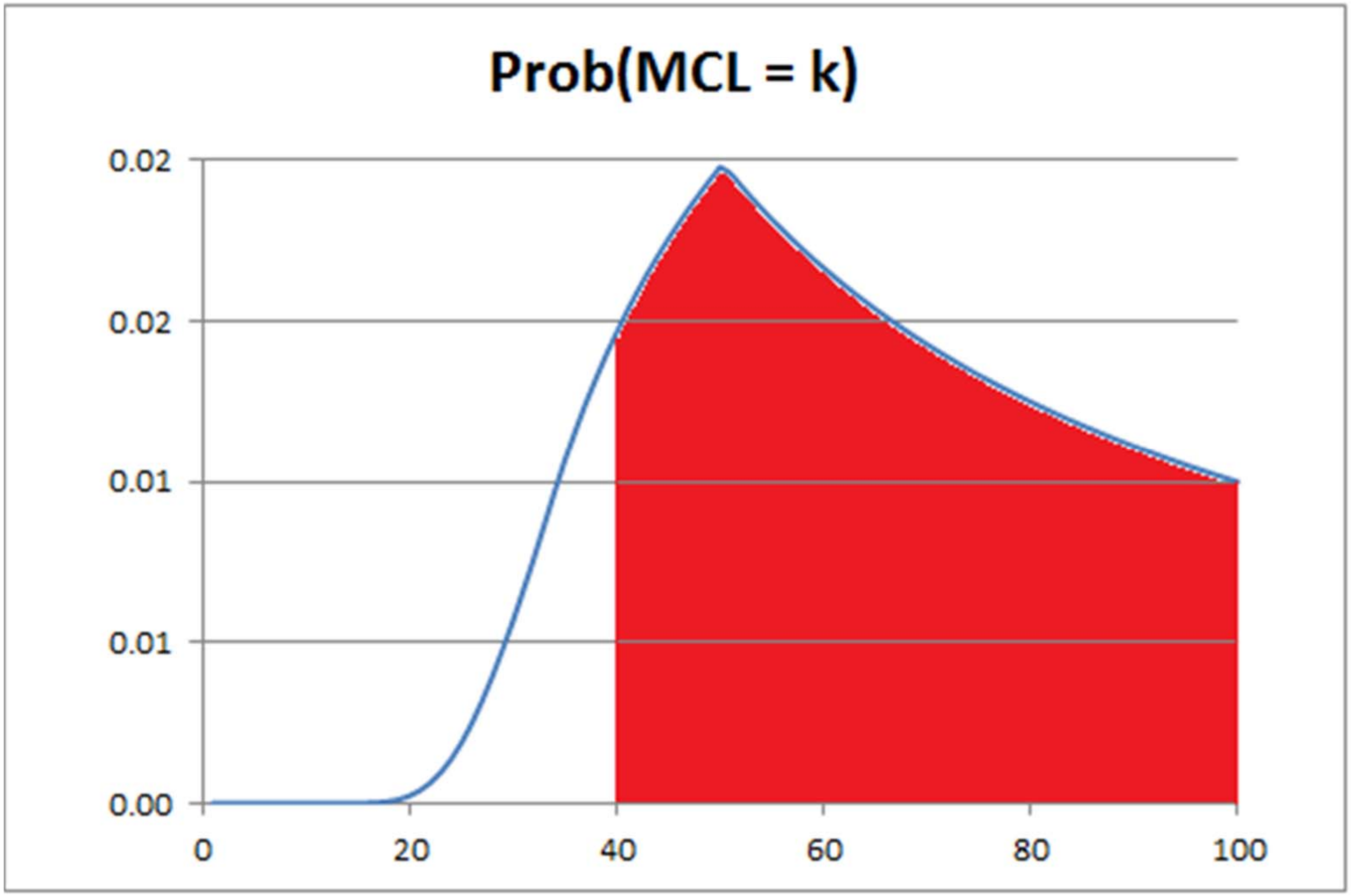
- Run through ALL possible partitions of 100 (there are 190,569,292 of them).
- For each one, calculate the probability that a random permutation has that cycle structure
- Sort these partitions according to max cycle length
- Add up the probabilities of all the partitions with a given cycle length *k*
- That is the probability that a random permutation has MCL = *k*

MCL Distribution for 100 Knights

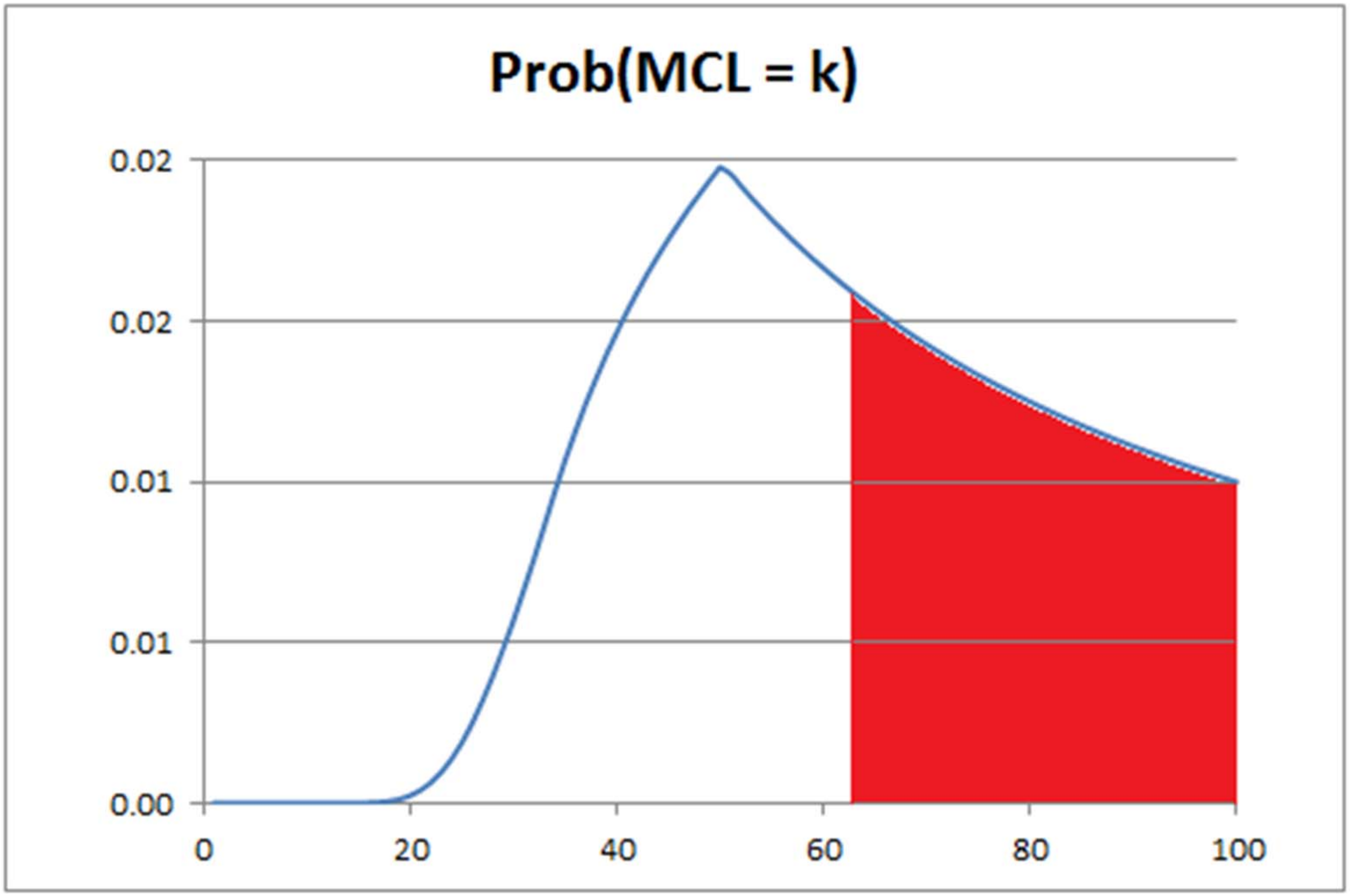


Game of Stones Opening 39 Boxes

All knights die if MCL $\geq$ 40



Game of Stones Opening 60 Boxes
All knights die if MCL $\geq$ 61



An Expectation Question

- Following the Sumwell's secret strategy, what is the expected number of knights who will succeed?
- Hint: Impossible for exactly 37 to fail, because each failure must be in a cycle of length at least 51.
- Surprise answer: E = 50.
- Same as for opening boxes at random.

Final Comments

- There are many interesting questions about MCL and related ideas
- A huge amount is known about these subjects
- Unlikely that there are any easy to find/prove new discoveries
- But this is a nice topic for recreational math