Elementary Mathematical Models and a Glimpse of Chaos

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Today's slides: **www.dankalman.net**
EMM resources:
www.dankalman.net/emm

Development Goals

- Intrinsic value/interest/significance
- Coherent story line
- Show power of algebra in context
- General education perspective on how math actually gets applied
- Decreased emphasis on abstract manipulative skills
- Highlight college algebra topics most likely to appear in client discipline introductory courses

Persistent Themes

- Discrete Sequential Data: $a_1, a_2, a_3, \cdots$ and approximating models; applying math through models
- Recursive patterns are easy to formulate: dependence of a_{n+1} on a_n. Difference equations
- Solutions to difference equations: explicit equation for a_n as a function $f(n)$; extension to continuous models
- Parameterized *families* of difference equations and solutions; fitting a model to actual data by choosing *best* values for parameters
- Direct prediction: evaluate $f(n)$ to predict data value number n
- Inverse Prediction: invert $f(n)$ to predict for which n the data value will reach a specified value
- Graphical, Numerical, and Theoretical methods

Arithmetic Growth

- Each term increases by fixed amount over preceding term
- Example: population grows by 5000 each year
- Difference equation $p_{n+1} = p_n + 5000$
- Solution: $p_n = p_0 + 5000n$
- Typical questions: What will the population be in year n? When will population reach 60000?
- In general: $a_{n+1} = a_n + d$; $a_n = a_0 + dn$
- Lead in to topic of linear equations

Quadratic Growth

- Computer network example. Adding one computer to a network of n requires adding n additional communications lines
- Difference equation $c_{n+1} = c_n + n$
- Solution: $c_n = n(n-1)/2$
- Typical questions: How many lines are needed for 200 computers? We have 500 lines, how many computers can we put on the network?
- In general: Each term increases over preceding term by an amount that depends linearly on n
- In general: $a_{n+1} = a_n + d + en$; $a_n = a_0 + dn + en(n-1)/2$
- Constant second differences
- Sums of arithmetic growth models
- Lead in to topic of quadratic functions and equations

Functional Equation Example

DIfference equation: $a_{n+1} = a_n + 22 + 10n$ with $a_0 = 5$

$n = 0$	5		
$n = 1$	$5 + 22$	$+$	$10 \cdot 1$
$n = 2$	$5 + 22 + 22$	$+$	$10 \cdot 1 + 10 \cdot 2$
$n = 3$	$5 + 22 + 22 + 22$	$+$	$10 \cdot 1 + 10 \cdot 2 + 10 \cdot 3$
$\vdots$			
$n = 6$	$5 + 22 + 22 + 22$	$+$	$10 \cdot 1 + 10 \cdot 2 + 10 \cdot 3$
	$22 + 22 + 22$	$+$	$10 \cdot 4 + 10 \cdot 5 + 10 \cdot 6$
$\vdots$			
n	$5 + 22 + \cdots + 22$	$+$	$10 \cdot 1 + \cdots + 10 \cdot n$

$$\begin{aligned} a_n &= 5 + 22n + 10(1 + 2 + \cdots + n) \\ &= 5 + 22n + 10n(n+1)/2 \end{aligned}$$

Geometric Growth

- Each term is a fixed multiple of the preceding term; equivalently, each term increases by a constant percentage over the preceding term
- Example: population doubles each year (increases by 100%)
- Difference equation $p_{n+1} = 2p_n$
- Solution: $p_n = p_0 2^n$
- Typical questions: What will the population be in year n? When will population reach 60000?
- In general: $a_{n+1} = ra_n$; $a_n = a_0 r^n$
- Lead in to topic of exponential functions and logs

Mixed Arithmetic and Geometric Growth

- Each term combines a fixed multiple of the preceding term with a fixed increment;
- Example: Pollution flows out of a lake in proportion to the existing concentration, but flows into the lake at a constant absolute rate
- Difference equation $p_{n+1} = .9p_n + 3$ (one tenth of the pollution flows out, and three more units are added, each unit of time)
- Solution: $p_n = p_0(.9^n) + 3(1 - .9^n)/(1 - .9)$
- Typical questions: What will the pollution load be in year n? When will it reach 100?
- In general: $a_{n+1} = ra_n + d$; $a_n = a_0r^n + d(1 - r^n)/(1 - r)$
- Realistic Applications: repeated drug doses, repeated loan payments or investments

Logistic Growth

- Modified version of geometric growth. Each term is a multiple of the preceding term, but the multiplier varies linearly with the size of the term

- Example: population p goes up in a year by a factor of $.01(200-p)$

- Difference equation $p_{n+1} = .01(200 - p_n)p_n$

- No explicit solution, but interesting qualitative behavior: initial growth similar to exponential, but levels off

- In general: $a_{n+1} = m(L - a_n)a_n$

- Interesting analysis results: $0 < mL < 1 \Rightarrow a_n \to 0$; $1 \leq mL < 3 \Rightarrow a_n \to L - 1/m$; $0 \leq mL < 4 \Rightarrow a_n \in [0, L] \forall n$;

Fixed Points

- Condition: $a_{n+1} = a_n$
- Logistic Growth: $a_{n+1} = [m(L - a_n)]a_n$
- Need $m(L - a_n) = 1$
- Fixed point $= L - 1/m$

Harvesting

- Diff Eqn: $a_{n+1} = m(L - a_n)a_n - h$
- Fixed Point equation

$$\begin{aligned} m(L - a_n)a_n + h &= a_n \\ m(L - x)x + h &= x \\ mx^2 + (1 - mL)x - h &= 0 \end{aligned}$$

- Generally two theoretical fixed points
- Fixed points key to analysis

Chaos in Logistic Growth Model

Logistic growth model can always be transformed to the quadratic family iteration by change of variable. This is a classical example in the study of chaos. In the context of real population model it is possible to introduce and explore concepts like:

- nonlinearity of the difference equation
- stability of the model under perturbations of parameters
- periodic orbits
- sensitive dependence on initial conditions
- bifurcation diagrams
- chaos