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Elementary Math Models: College Algebra Topics and a Liberal Arts Approach

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Introduction

It would be difficult to overstate the importance of algebra in mathematics. If mathematics is the language of science, then algebra provides the alphabet, vocabulary, and syntax. In particular, the traditional college algebra course covers the elementary functions of analysis: linear and quadratic functions; polynomials and rational functions; roots, exponentials, and logs. These functions are inescapable in the most elementary applications of mathematics to other subjects. In a word, they are ubiquitous all over the place.

It is tempting, therefore, to prescribe algebra as the minimal quantitative component of a higher education. Unfortunately, for many students who study algebra in college, their mathematical education goes no further. For these students, the significance of algebra is largely lost. It is as if they studied an alphabet, vocabulary, and syntax, but never got to read any literature.

Liberal arts math courses offer an alternative exposure to mathematics for general education students. These courses emphasize lofty educational goals, hoping to communicate something of the beauty, power, and fascination of our discipline. Developing specific manipulative skills is given much less emphasis.

This essay concerns a hybrid of these two approaches to general education mathematics. Elementary Mathematical Models (EMM) was developed in the mid 1990s at American University, a selective medium size university in Washington, DC. It is offered at the lowest level of the mathematics curriculum and fulfills the general education mathematics requirement. Like a traditional college algebra course, EMM seeks to make students familiar with the elementary functions. But like a liberal arts math course, it strives for an intrinsic educational significance independent of utility for any other courses.

Classroom materials for EMM evolved into a textbook, now available from the MAA [1]. Additional resources are available on the web [2]. The course has been used successfully for about five years at American University and a few other institutions. In the discussion below, I will provide an overview of the course and its clientele as I observe them at American University, concluding with a summary of student reactions and performance.

Content and organization

The core of the EMM course is a sequence of progressively more complicated growth models: arithmetic growth, quadratic growth, geometric growth, mixed models, and logistic growth. The first three of these give rise to important families of elementary functions, linear, quadratic, and exponentials, respectively. Logs are introduced as part of the material on exponential functions. Mixed models, which are a combination of arithmetic and geometric growth, give rise to shifted exponentials of the form $Ae^{bt} + c$. This core material can be expanded in a variety of ways, covering closely related units on polynomials and rational functions, additional properties of logs, linear regression, or chaos.

In EMM, mathematical topics are always introduced in the context of some realistic modeling problem, and the mathematical exposition never strays far from the applied context. Algebra is presented when and where it is needed and in conjunction with numerical and graphical methods. I made a commitment in developing the course to omit any topic or skill that could not be immediately justified in the context of application. That is, I only included topics that students could see they needed. It should not be surprising that relatively little of the traditional material on elementary functions had to be discarded in this way. Afterall, those topics are in the traditional curriculum because they *are* useful. EMM simply exploits this fact by developing each topic within the context where it is needed. EMM students never ask, *Why is this topic useful?*

By design, EMM incorporates themes that appear repeatedly throughout the course. Acquiring a deep understanding of these themes and retaining that understanding beyond the end of the course are among the liberal arts goals of the course. Some of the themes are methodological and procedural, others are more philosophical. For example, the modeling framework as a broad methodology for applying mathematics constitutes a philosophical theme. I hope students will, by the end of the semester, understand that models involve simplifying assumptions; that there is a spiral aspect to model development, evaluation, and refinement; that mathematical analysis provides a powerful tool for tracking the consequences of simplifying assumptions; and that mathematical models have both strengths and limitations.

A more procedural theme concerns the use of discrete models with simple recursive patterns. The general framework is a sequence of data values or predictions, and the recursive pattern specifies how each successive term is obtained from its predecessor. In arithmetic growth, the recursive pattern is addition of a constant; in geometric growth it is multiplication by a constant. These patterns are expressed as difference equations. For example, $a_{n+1} = a_n + 2$ describes an arithmetic growth pattern where each term is found by adding 2 to the preceding term. With the exception of logistic growth models, all the simple difference equations have corresponding solutions, which express the terms of a sequence, a_n, as a function of n.

As part of this theme, there is a repeated pattern of development for each family of growth models. In each topic, we look at several examples of different phenomena (e.g. metabolization of drugs, pollution in a body of water, repeated loan payments) and find a simple recursive pattern that is common to all of them. The pattern is expressed as a specific kind of difference equation. Systematic exploration of these difference equations leads to solutions comprising a family of elementary functions. Graphical, numerical, and analytic properties of these functions then reveal information about the evolution of the corresponding models.

EMM makes no attempt to survey modeling in a broad way. In fact, EMM is not a modeling course at all. Its primary goal is not to teach students all about mathematical modeling. Rather, models of a very particular kind, discrete recursive models, are used as a vehicle for motivating the study of elementary functions. In the process, I hope students will acquire a realistic sense of how mathematics is truly applied and experience some of the creativity and judgement that goes into applying mathematics to real problems.

Students

The students of EMM have as a minimum the traditional college preparatory mathematics coursework at the secondary level. This typically means something like three years of high school math, often algebra 1, algebra 2, and geometry. However, the specific algebraic skills of my students can be quite low. Presumably, this reflects both imperfect mastery of material in the earlier courses, as well as attenuation over time of whatever skills were originally learned. In fact, students who master and retain three years of high school mathematics would probably not be placed in EMM at American University.

Part of the design of EMM was a reaction to working with students for whom abstract symbol manipulation is nearly meaningless. I see these students often. Typically they are willing to work diligently in the course. They also often have moderate to good number sense and can function quite well in the concrete context of a particular problem. Somehow, though, the language of algebra just does not work for them.

This point is worth elaborating. One of the advantages of algebraic notation is its ability to capture numerical relationships or patterns in a very general, yet very succinct way. Take for example, the difference equation $a_{n+1} = 3a_n$. To anyone with a command of algebra, this says in a very compact way, each term is three times the preceding term. I regularly see students to whom it says nothing of the kind. These students easily apprehend and work with the pattern of successive tripling of terms in a sequence. When that sequence is part of a model, they can handily use numerical methods to answer questions about the model. In short, they understand everything about the model that I want them to understand. But, whereas to me the equation captures most or all of that understanding, for these students, the equation does not. They cannot quite see what the point of the equation is and can only translate between the abstract notation of the equation and their own clear understanding of the pattern with great effort and concentration.

For these students, the context of a particular problem or family of problems is critical. They cannot attach any meaning to the symbolic notation otherwise. EMM tries to assist these students by keeping the mathematics firmly in a meaningful context. I hope students will use their concrete number sense, general reasoning ability, and verbal skills to help them endow the symbolic notations with meaning.

I do not want to leave the impression that EMM students *all* struggle with algebra. On the contrary, the audience is quite diverse, and frequently one or two of the students in EMM have even completed a semester of calculus. For the strongest students, the contextually thick buildup to each symbolic abstraction can be a bit annoying. But overall, these students do not find the course boring or trivial. The realistic models make the material interesting, and the constant connections between mathematical procedures and application issues is something that is unfamiliar to just about all of the students.

Classroom practice

The EMM course does not presuppose any particular style of presentation. When I teach the course, I use a mixture of lecture-discussion, small group activity, and computer exercises. Some of my colleagues have used mainly traditional lecture style presentation. The course is intended to be language rich, and reading and writing are heavily emphasized.

The application contexts and recursive models lend themselves nicely to empirical investigation. This is easily supported by technology in a number of ways. I have successfully used graphing calculators and special purpose computer activities designed for the course. These were implemented in an authoring package called Mathwright [4] and are freely available over the internet [2]. They have the interaction style of a webpage, allowing students to enter data and equations in a natural and intuitive way and explore graphical and numerical properties of models. It is also possible to provide similar kinds of computer activities using a spreadsheet program, such as Excel. I imagine that notebooks in Mathematica or Maple could also be created, but I have not done so.

Sample lesson

To illustrate the evolution of ideas in the course, here is an outline for one topic, mixed models. This material would be covered in about one and one-half weeks. For additional samples, see [3]. Mixed models are characterized as a combination of geometric growth and arithmetic growth and by a difference equation of the form:

$$a_{n+1} = ra_n + d$$

with r and d fixed constants. These models arise in a variety of contexts, including:

- pollution in a body of water
- metabolization of medicine with repeated doses
- amortized loan payments
- sums of geometric growth models

Given the formula $1+r+r^2+\cdots+r^{n-1} = (r^n-1)/(r-1)$ for the sum of a geometric series, a simple pattern analysis leads naturally to the solution of a mixed model difference equation. This is presented in terms of a numerical example of the following sort. Consider the sequence a_n, where $a_0 = 200$, and each successive term is 3/4 of the preceding term, plus 100. We systematically generate several terms of the sequence, without actually carrying out any of the arithmetic:

$$\begin{aligned} a_0 &= 200 \\ a_1 &= 200(.75) + 100 \\ a_2 &= 200(.75^2) + 100(.75) + 100 \\ a_3 &= 200(.75^3) + 100(.75^2) + 100(.75) + 100 \end{aligned}$$

Here, the right-hand side of each equation is obtained by multiplying every term on the preceding line by .75, and then appending an additional increment of 100. Students quickly recognize the pattern and are soon led to discover

$$a_n = 200(.75^n) + 100\frac{1-.75^n}{1-.75}.$$

It is a small step from this result to the generalization:

$$\text{if } \; a_{n+1} = ra_n + d, \quad \text{then} \quad a_n = a_0(r^n) + d(1-r^n)/(1-r).$$

Students study the graphs and the numerical tables for these sequences and observe that for $r < 1$, the values of a_n level off to an equilibrium value. In one lab period, they explore a model for repeated drug doses numerically and graphically, discovering that the equilibrium value does not depend on the initial dose of medication, but is proportional to the size of the repeated dose and inversely proportional to the percentage of the drug which is eliminated from the body between doses. They use these observations to determine the size of the repeated dose required to achieve a predetermined equilibrium level of medication retained in the body.

The algebraic lessons associated with this unit focus on algebraic rearrangement and solving equations. One of the goals of the course is to make it clear to students why algebraic rearrangement is necessary and useful. In this unit, they see that the natural form for a solution to a difference equation, for example:

$$a_n = 200(.75^n) + 100\frac{1-.75^n}{1-.75},$$

can be expressed more compactly in the form:

$$a_n = 400 - 200(.75^n).$$

Each form has value in some context. The first form shows clearly the significance of the parameters a_0, r, and d. The second is more convenient for computation and shows at once that the graph is a vertically shifted exponential, with equilibrium value 400. In this way, the abstract practice of algebraic rearrangement is observed to have significance in a very concrete way.

A second repeated appearance of algebra occurs in connection with inverting the functions which arise as solutions to difference equations. In the context of the example discussed above, we might ask when the amount of drug will reach a value of 350. Using the simplified form of the equation for a_n, that leads to:

$$350 = 400 - 200(.75^n).$$

Here, algebraic rearrangement is again used, this time to reduce the equation to one which the students can solve with logs:

$$.25 = .75^n.$$

In this unit, there is also a discussion of fixed points and their role in determining equilibrium values. If x is fixed by the recursive operation *multiply by .75 and add 100*, then it follows that:

$$.75x + 100 = x.$$

This provides an alternative route to the equilibrium value of 400.

Assignments for this unit include applications of mixed models that arise naturally in all of the areas cited above. Students can formulate models for these application areas and use numerical, graphical, and symbolic methods to predict future behavior of the models. They also work with the idea there are several different justifications for using a mixed model (or any other kind of model). In some of the applications, simple assumptions about mechanisms at work in the model (like elimination and repeated ingestion of medications) lead to a mixed model. In other cases, the data are simply observed to fit closely to a mixed model pattern. In yet other situations, a previously defined geometric model is summed to obtain a mixed model. For example, a geometric growth model for annual oil consumption leads directly to a mixed model for world petroleum reserves.

Student reactions and performance

Students have been very supportive of the goals and framework of the EMM course. Between 80% and 90% describe the course as interesting and worthwhile. Each semester, there are a few students (perhaps 3 or 4 in a class of 30) who make comments like these:

> *Best math course I ever took; I am usually awful at math but I really understood this course; first time I actually enjoyed a math course; I was dreading math but this turned out to be one of my favorite courses.*

There also are always a few students who object to the emphasis on writing and *thinking*. It is rare for a student to find the material so easy that it offers no intellectual challenge. On the other hand, the stronger students would find that a traditional college algebra course can be completed in a much more mechanical way and with much less in the way of conceptual demands. Perhaps these are the same students who complain: *Too much writing. I never had to write essays in a math class before.*

In my classes, almost every student who makes a reasonable effort over the course of the semester completes the course successfully, ending with a grade of C or better. Roughly half to two thirds of my students receive final grades of B or above.

I have not gathered any data on the long term instructional goals that provided a primary motivation for creating this course. I do not know how well EMM students retain ideas about mathematical models in general, about how and why math is applied, about the specifics of recursive patterns and functions. To that extent, I cannot really substantiate how effective the course is.

Conclusion

As a teacher, I find the EMM course very satisfying. It has a coherent story line and an evolving conceptual thread that stretches from the simplest models to current concepts in chaos. There are well-defined long-term instructional goals, and the repeated emphasis of aspects of these goals enable students to make progress through the entire semester. I am confident that the course is laying a foundation for the quantitative demands of other general education courses in the natural and social sciences. But at the same time, I feel that EMM students have an opportunity to learn something significant about how math is used, and why math is important. In addition, positive student reactions reinforce the philosophical convictions that inspired the course to begin with.

References

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