The Most Marvelous Theorem in Mathematics

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Outline

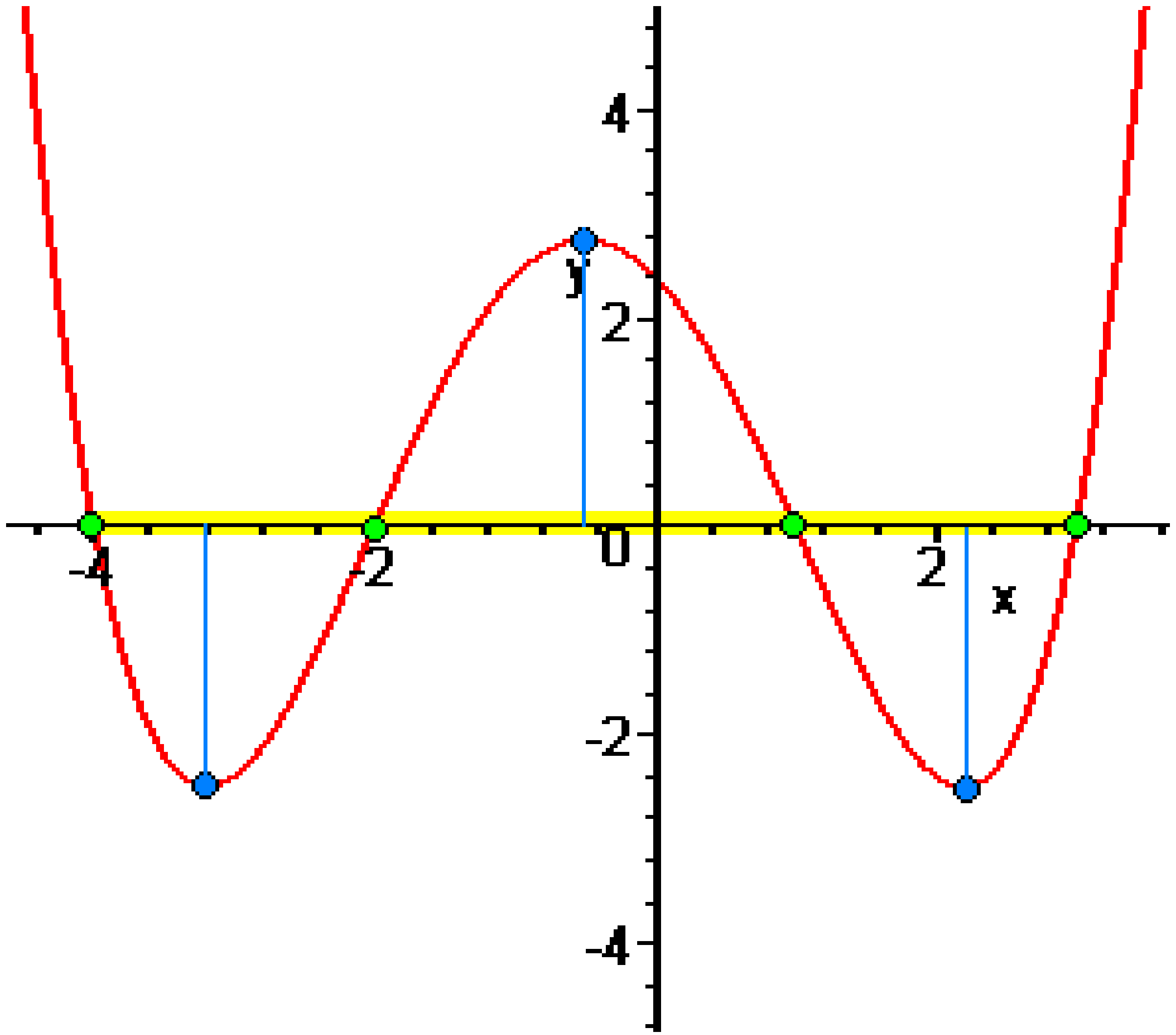
- Overview of the theorem
- Ellipses
- Background Facts
- Proof

Real Polynomials

- Familiar functions: $4x^5 - 3x^4 + x^3 - 5x^2 - 7x + 3$
- Whole number exponents
- Real coefficients
- No squareroots, *x*'s in denominator, *named* functions

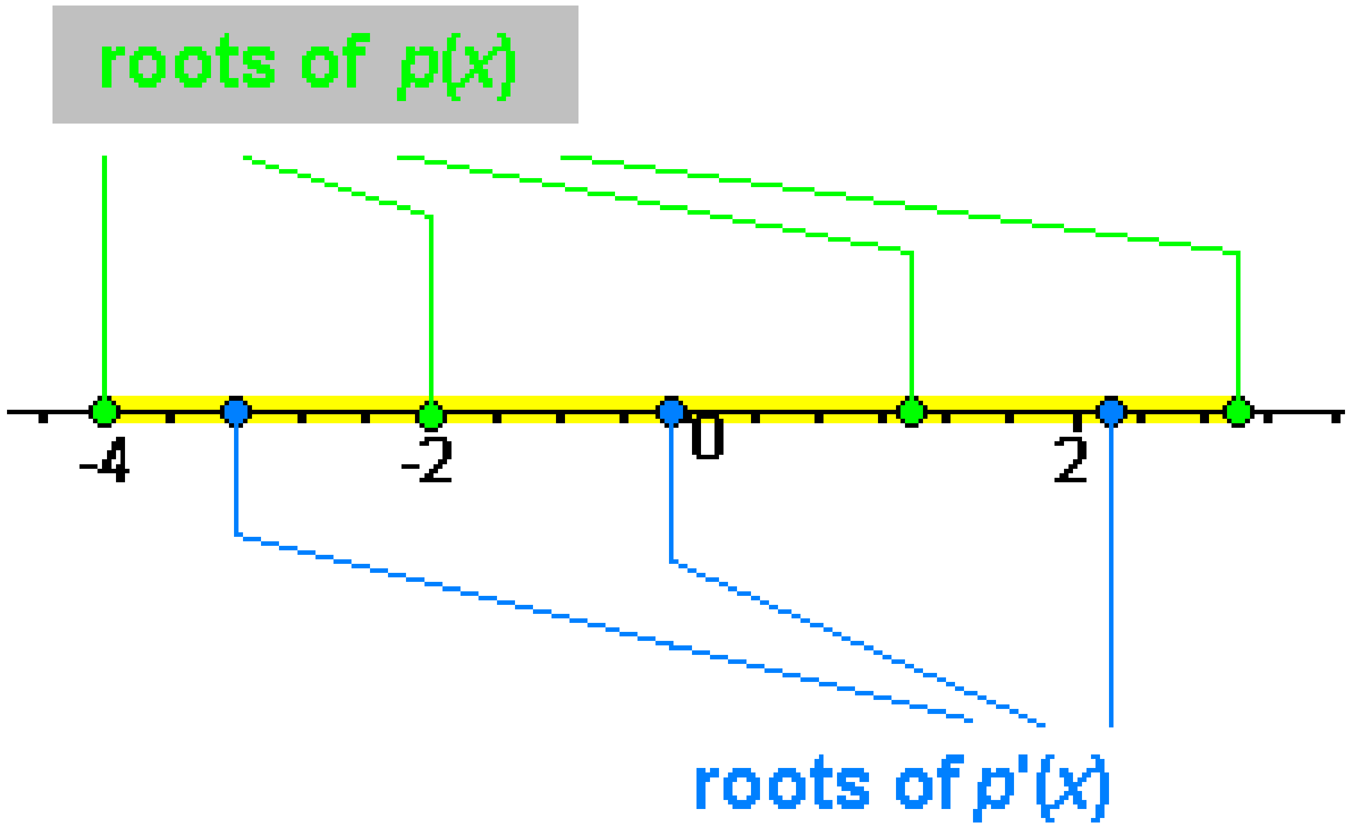
Real Polynomials with all Real Roots

- Roots of $p'(x)$ interlace roots of $p(x)$
- All roots of $p'(x)$ in the interval between least and greatest roots of $p(x)$

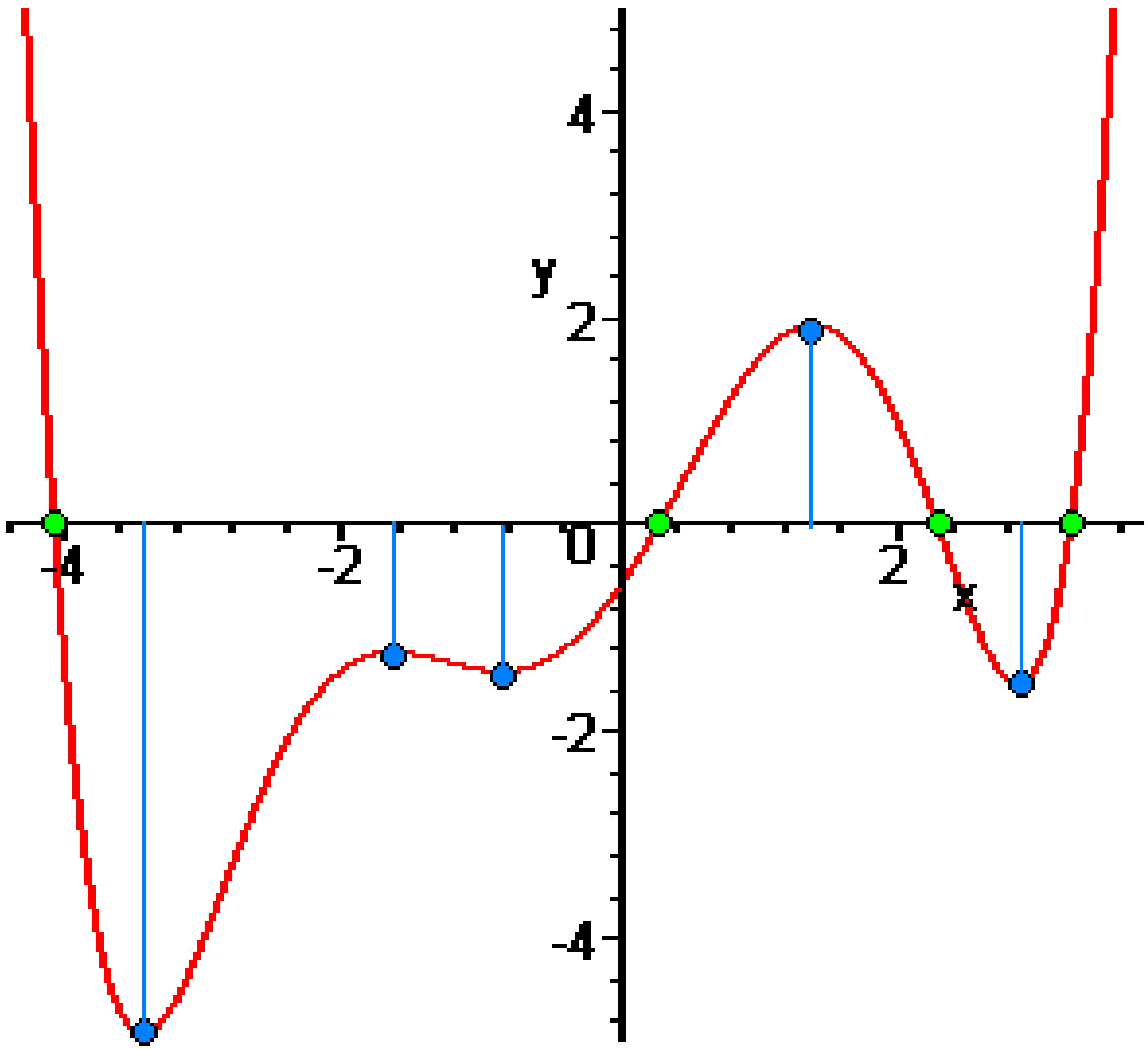


One Dimensional View

- Just view domain of *p*(*x*)
- Identify special points with labels



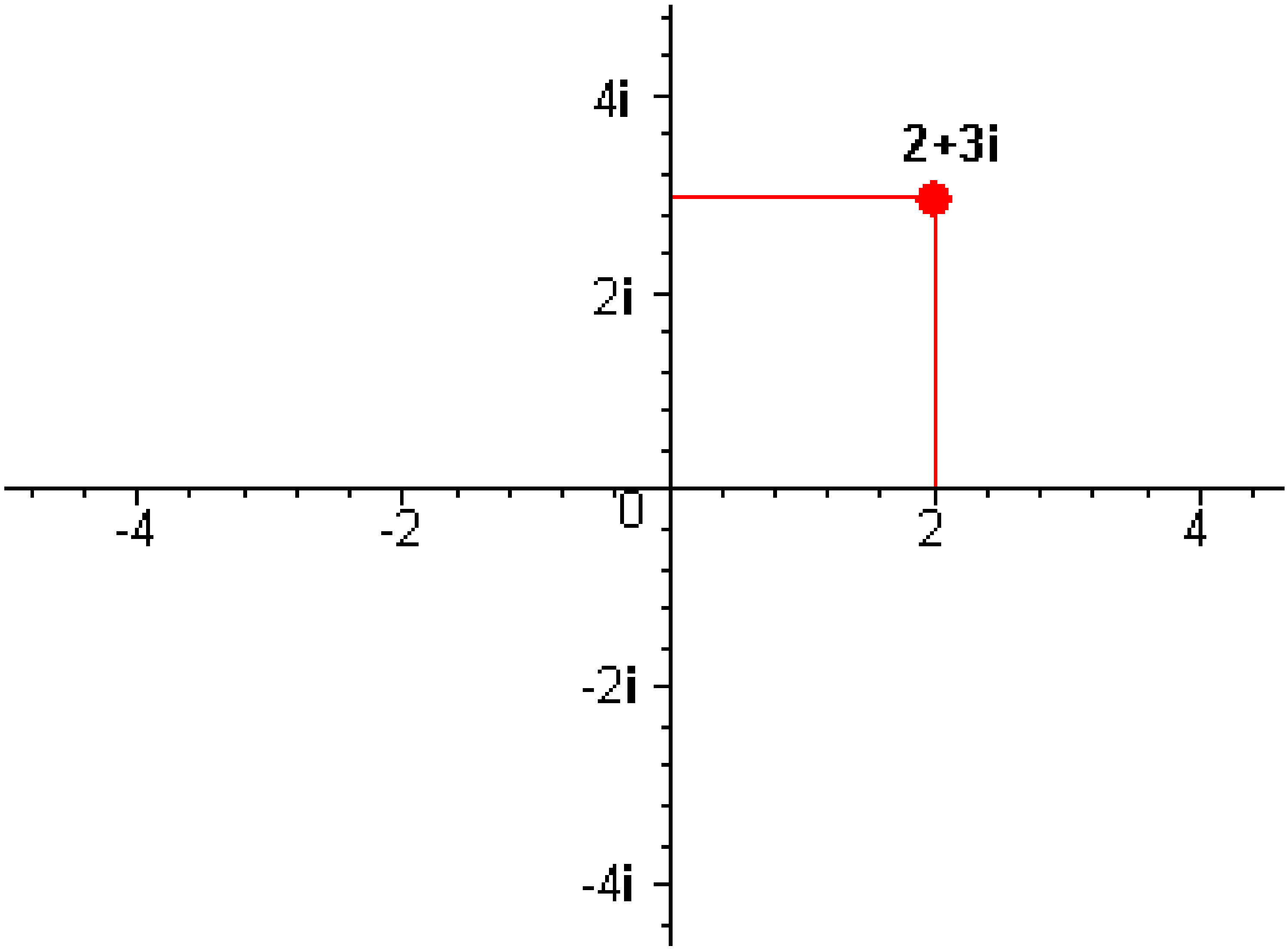
Complex roots …?



Complex Numbers

- i = square root of -1
- $i^2 = -1$
- General complex form $x + iy$
- Example 2+5i
- Add, subtract, multiply, and divide like regular numbers
- Picture as points in a plane (as opposed to a line)

Complex Plane



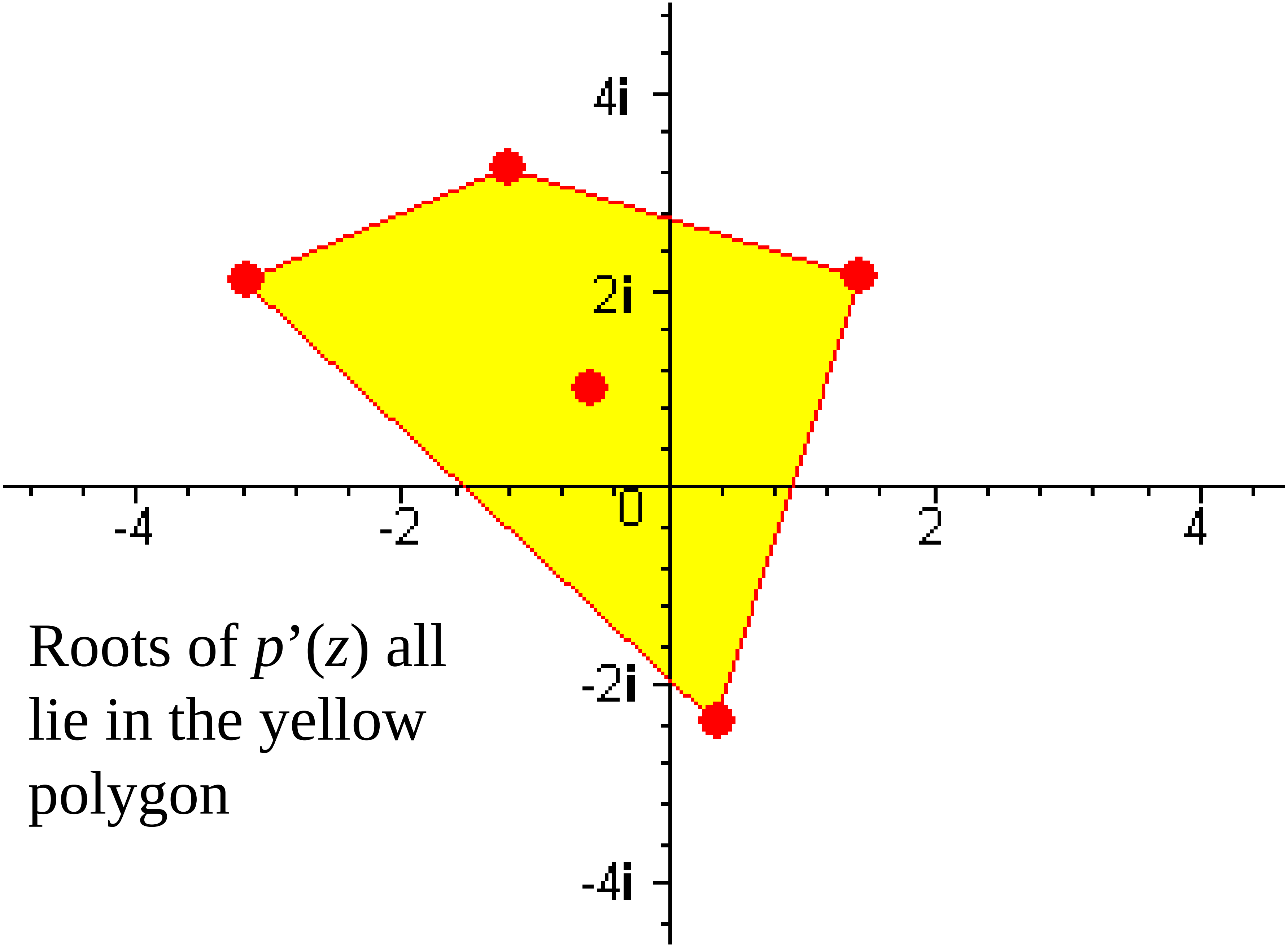
Complex Polynomials

- Similar familiar functions $4z^5 - 5z^2 - 7z + 3$
- Variable can be replaced with real or complex values
- Coefficients can be real or complex
- The domain is graphed as a plane – the complex plane

Complex Poly Facts

- Every complex polynomial factors to something of the form
$$\alpha(z - z_1)(z - z_2)(z - z_3)\cdots(z - z_n)$$
- For degree (highest exponent) *n*, you get *n* roots (z_1, z_2, etc.)
- Double count repeated roots as necessary
- Lucas's Theorem: the roots of $p'(z)$ all lie in the convex hull of the roots of $p(z)$.

Lucas' Theorem

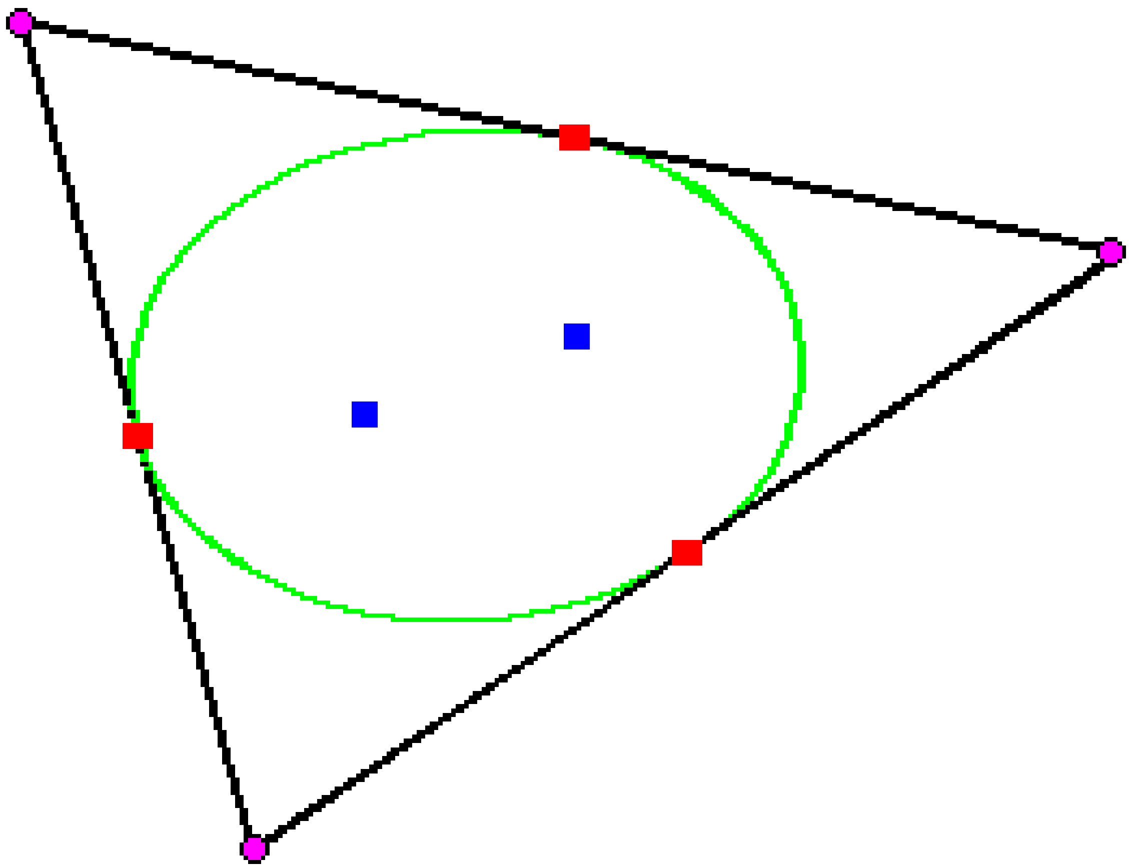


Roots of *p*'(*z*) all lie in the yellow polygon

Marden's Theorem

- Special case: cubic polynomial $p(z)$
- Roots are 3 noncolinear points in complex plane
- Convex hull is a triangle
- Where (exactly) are the roots of $p'(z)$?
- At the foci of an ellipse inscribed in the triangle

- Show roots of *p*(*z*)
- Show triangle
- Bisect sides
- Inscribe ellipse
- Mark foci
- Those are the roots of *p*'(*z*)
- INCREDIBLE



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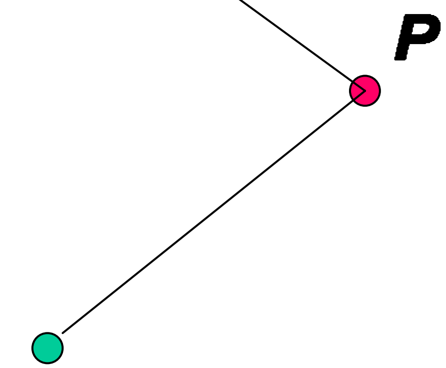
Ellipse Definition

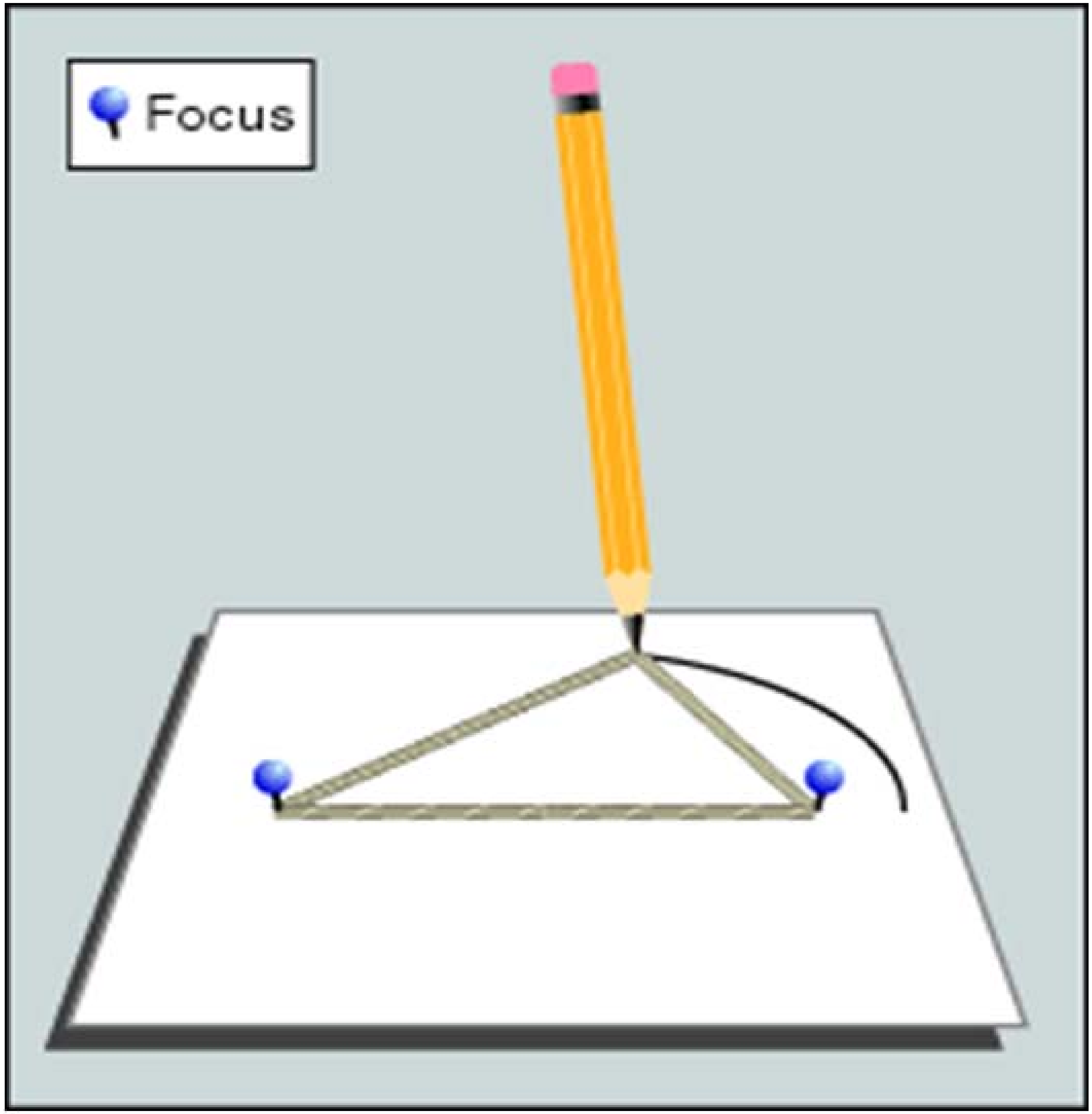
- Let two points be given *E* and *F*
- These are called *foci* of the ellipse

  (1 focus, 2 or more foci)

  (faux sigh)
- Let a fixed distance *d* be given
- *P* is a point of the ellipse if and only if traveling from *E* to *P* and then to *F* has combined distance *d*.

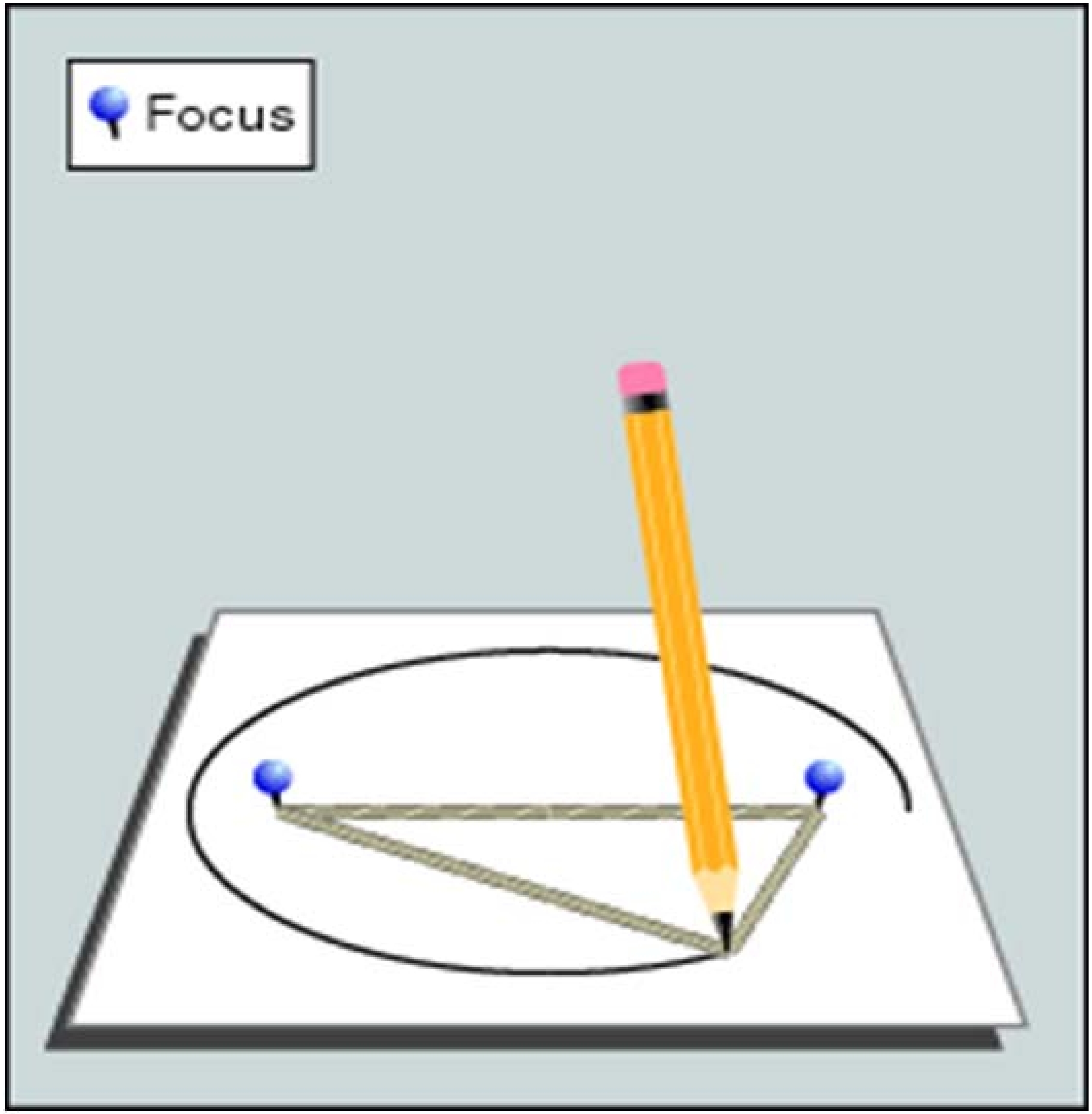


Pencil,
Pins, String



http://ccins.camosun.bc.ca/~jbritton/ellipseanim.gif

Pencil,
Pins, String



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An Ellipse Fact

- Let points *E* and *F* be given.
- Let point *P* be given.
- There is a unique ellipse with foci *E* and *F* going through point *P*
- Just find the distance from *E* to *P* and then to *F*. That becomes the fixed distance defining the ellipse

Animating Marden's Theorem

- Interesting fact: the roots of $p(z)$ and $p'(z)$ have the same average
- For cubic, roots of $p(z)$ are vertices of a triangle; average is the centroid
- This has to be halfway between the two roots of p'
- If you know where one root of p' is, then you know where the other is

Animating (cont.)

- Draw the triangle.
- Guess where one root of *p*' is
- That determines where the other root would have to be
- Use these as foci
- Make an ellipse with these foci going through midpoint of one side
- Repeat for other two sides
- Hope that all three ellipses coincide, and are tangent to the sides at the midpoints

- Click one point – possible root
- Reflect through centroid to define other root
- Draw ellipse with these foci and passing through the midpoint of one side
- Repeat for other two sides
- Drag the clicked point around until all three ellipses coincide
- The ellipses become tangent to sides at midpoints
- The foci are roots of *p*’

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Four Background Facts

- One polynomial fact
- One Complex Number Fact
- Two Ellipse Facts

A polynomial fact

- Quadratic Polynomial $Az^2 + Bz + C$
- Roots r and s
- *Fact:* $rs = C/A$
- Proof: Factor polynomial

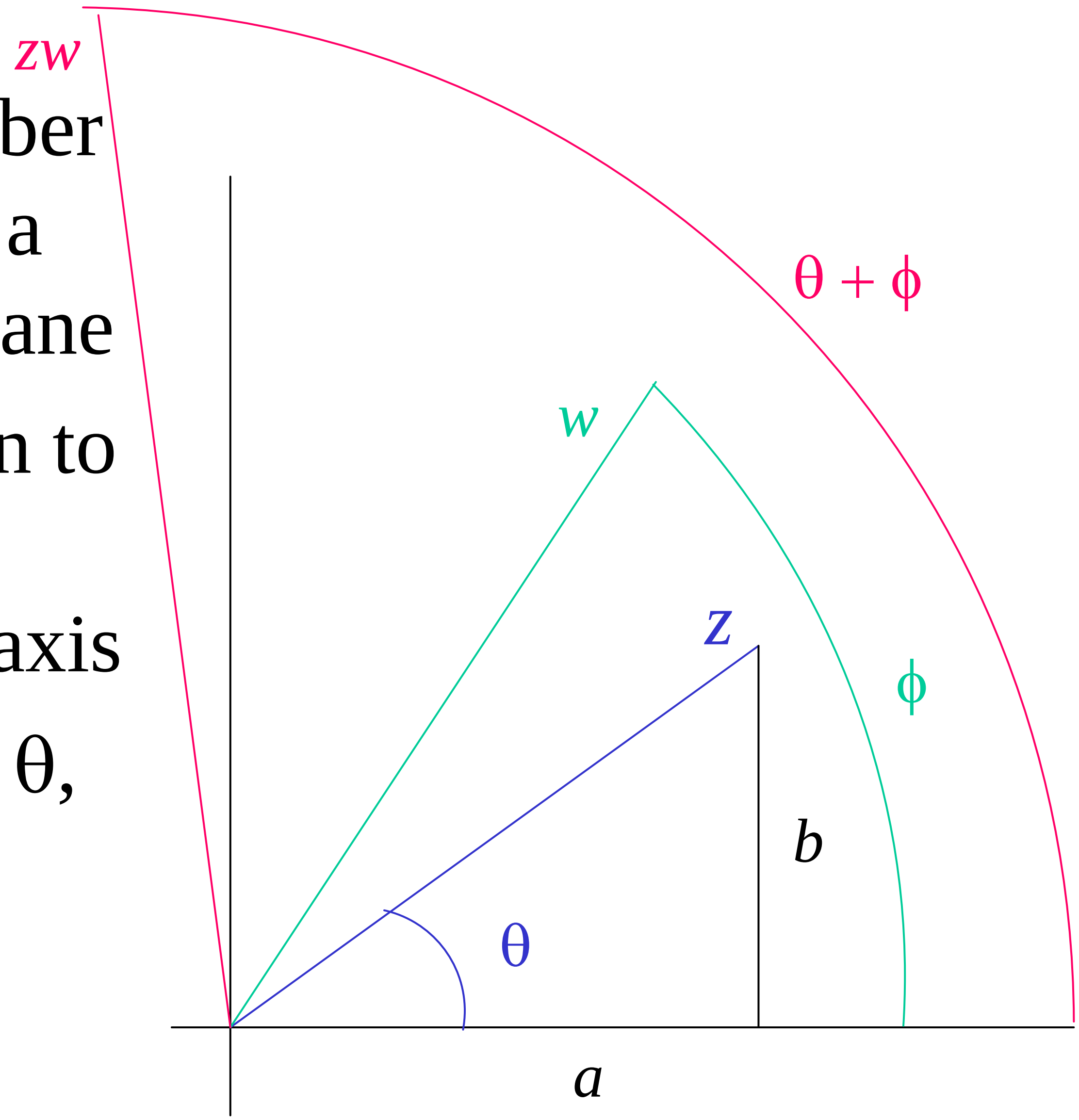
$$Az^2 + Bz + C = A(z^2 + B/A\, z + C/A\,)$$

$$= A(z - r)(z - s)$$

$$= A(z^2 - (r+s)\, z + rs\,)$$

A Complex Number Fact

- Each complex number $z = a+bi$ appears at a point (a,b) in the plane
- The line from origin to z makes an angle θ with the positive x axis
- *Fact*: If z has angle θ, and w has angle ϕ, then zw has angle $\theta + \phi$



Proof of Complex Fact

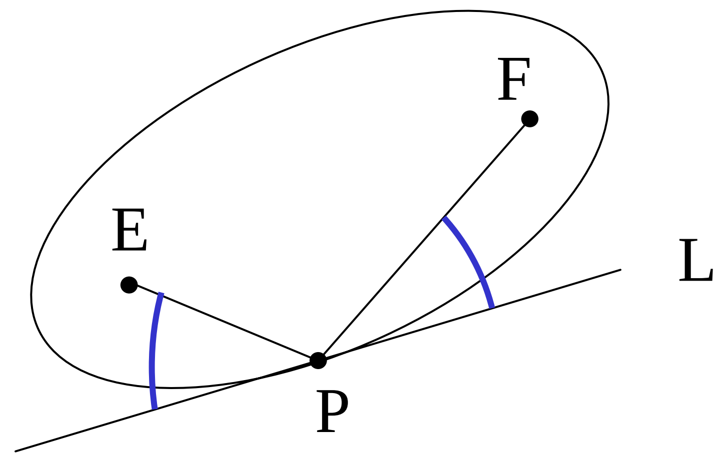
- $z = a + b\mathrm{i}$, $\tan \theta = b/a$
- $w = c + d\mathrm{i}$, $\tan \phi = d/c$
- $zw = (ac-bd) + (ad+bc)\mathrm{i}$
- $\tan(\theta + \phi) = (\tan\theta + \tan\phi)/[1 - \tan\theta \tan\phi]$

  $= (b/a + d/c) / (1 - bd/ac)$

  $= (bc + ad) / (ac-bd)$

Ellipse Fact 1

- Line L is tangent at point P to an ellipse with foci E and F
- *Fact*: Lines EP and FP make equal angles with line L
- If and only if: equal angles implies tangency

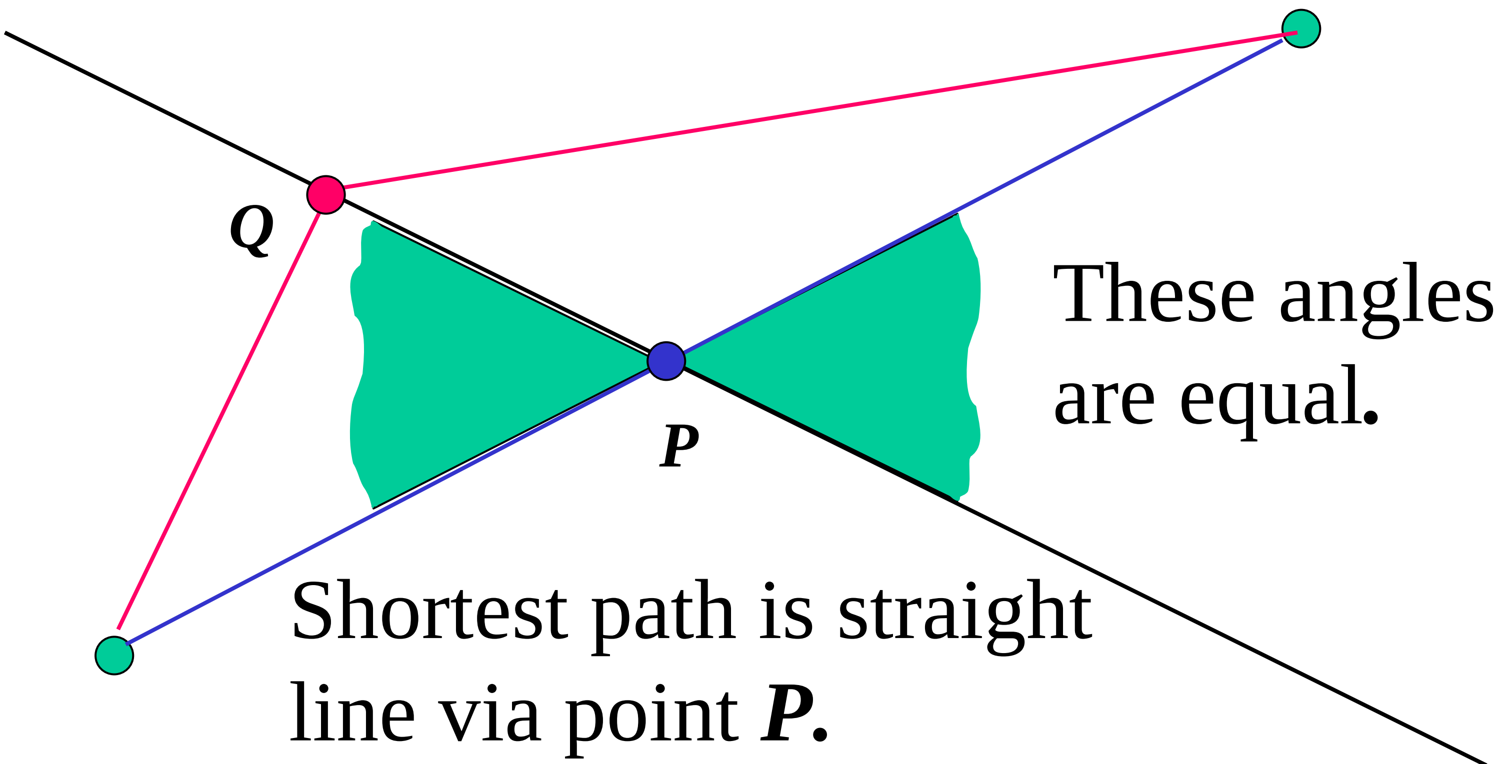


The distance E-P-F is smaller than for any other point on line L. No other ellipse with the same foci can be tangent to L.

Shortest Path: Point →Line→Point

Line Separates Points

Two line path via any other point ***Q*** is longer

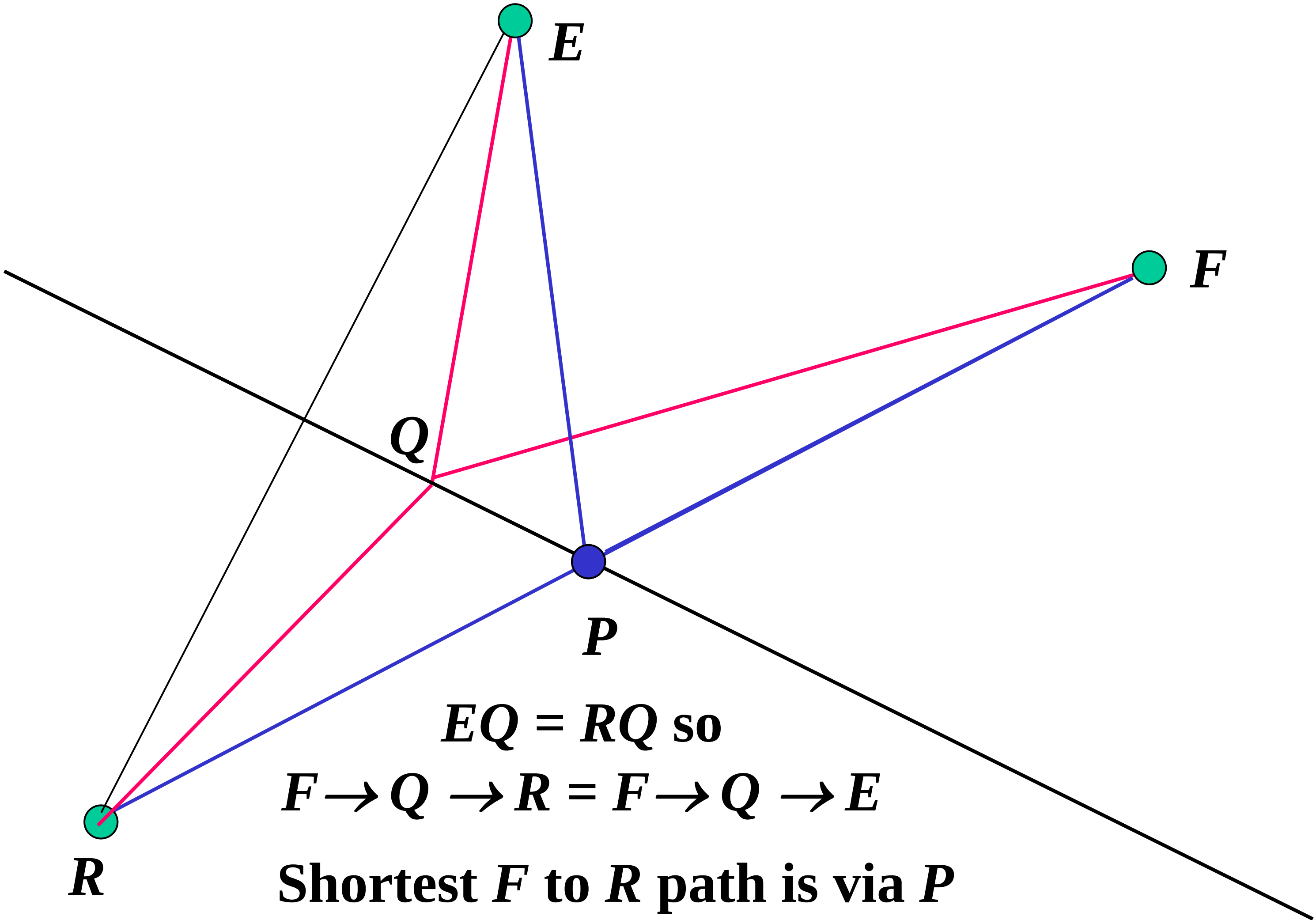


These angles are equal.

Shortest path is straight line via point ***P***.

Shortest Path: Point →Line→Point

Line Does Not Separate Points



***EQ* = *RQ* so**

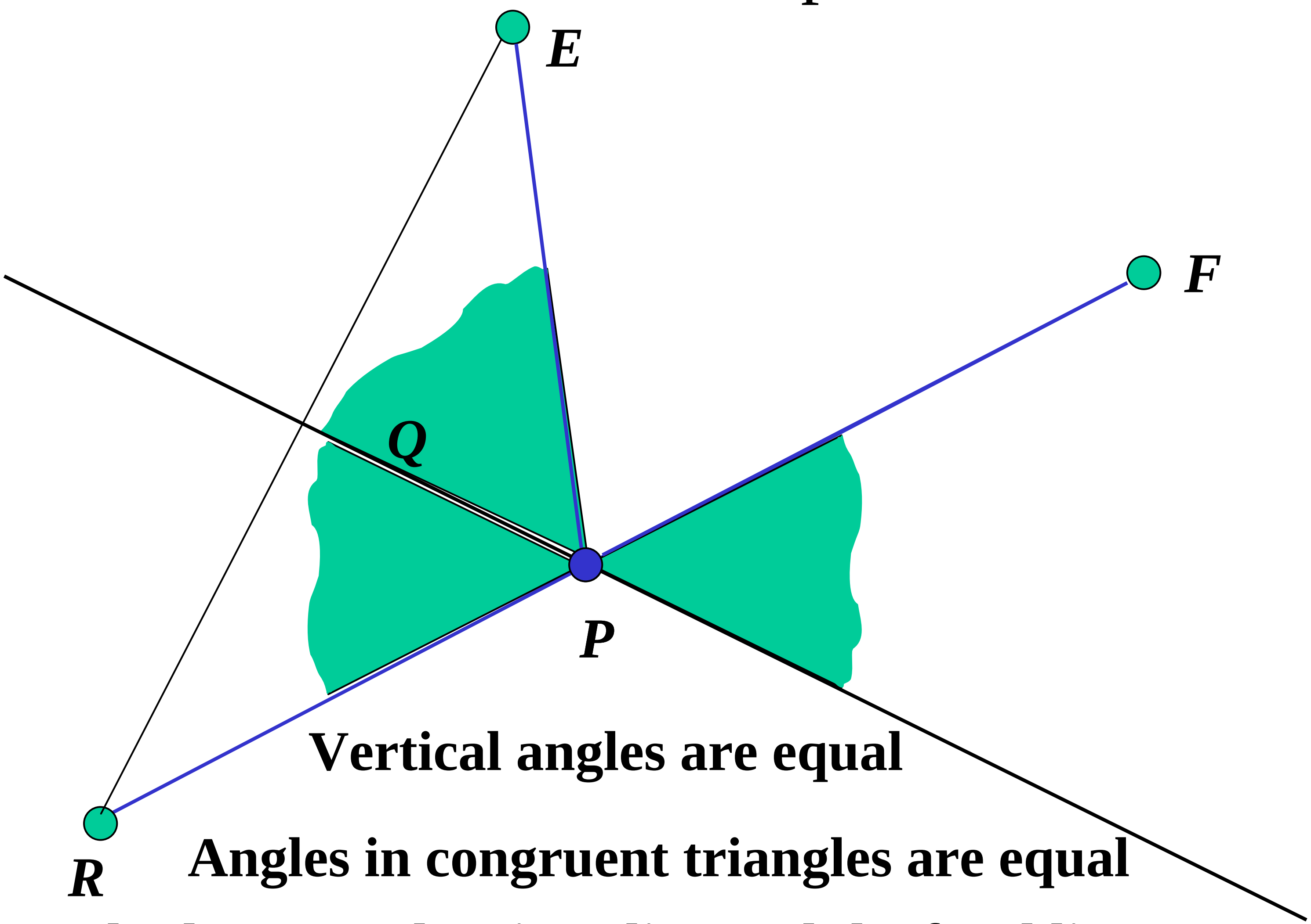
$$F \to Q \to R = F \to Q \to E$$

Shortest *F* to *R* path is via *P*

So shortest *F* to *E* path is via *P*

Shortest Path: Point →Line→Point

Line Does Not Separate Points



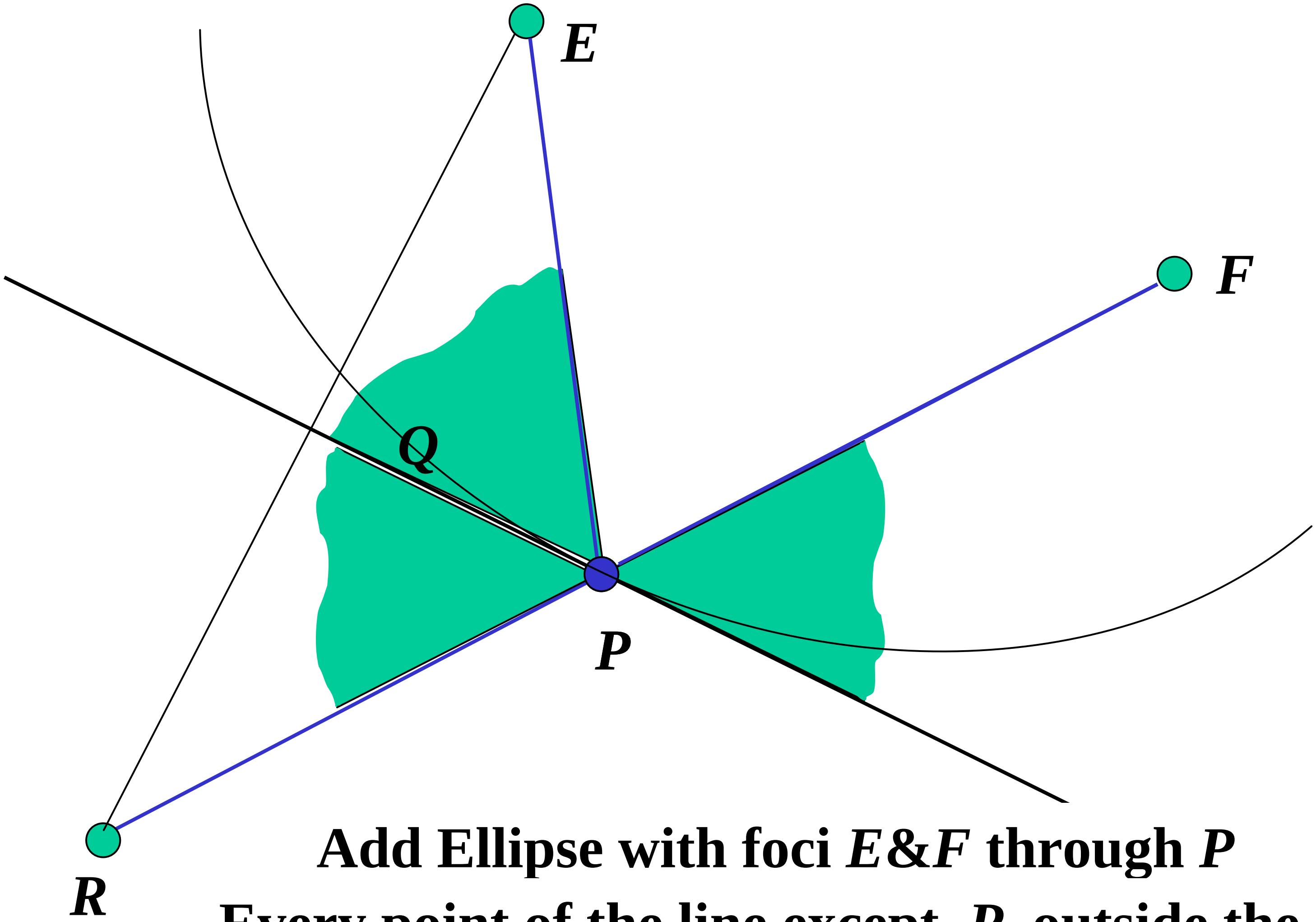
Vertical angles are equal

Angles in congruent triangles are equal

Angles between the given line and the focal lines are equal

Shortest Path: Point →Line→Point

Line Does Not Separate Points



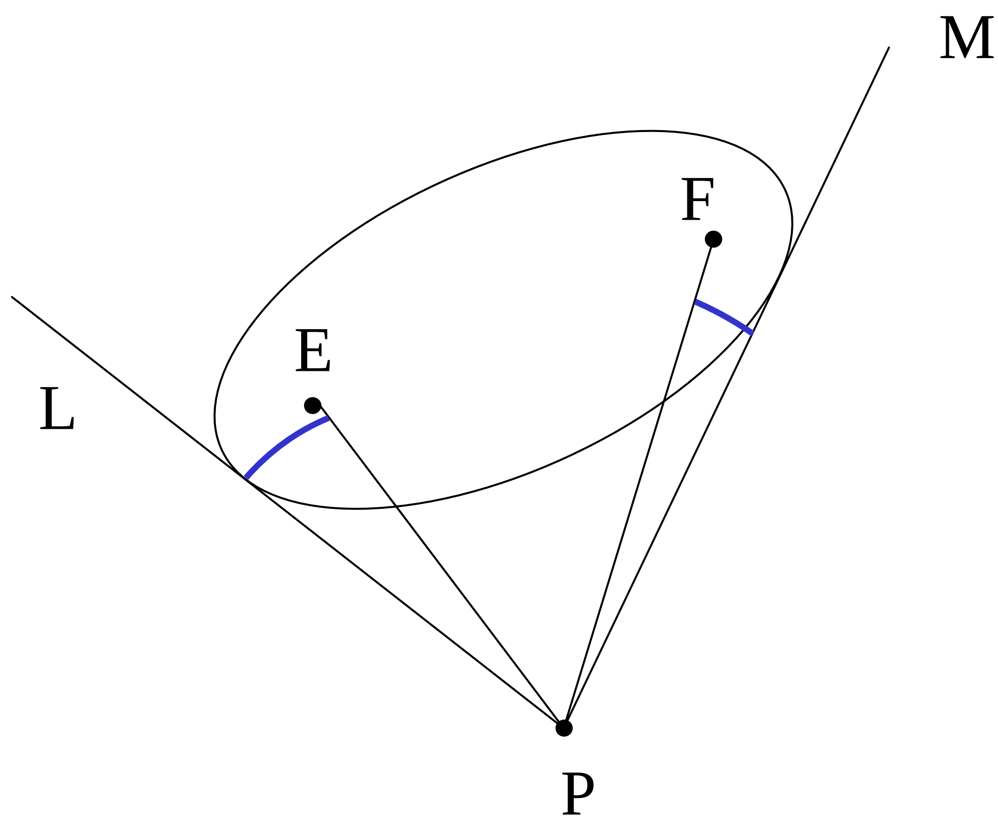
Add Ellipse with foci *E&F* through *P*

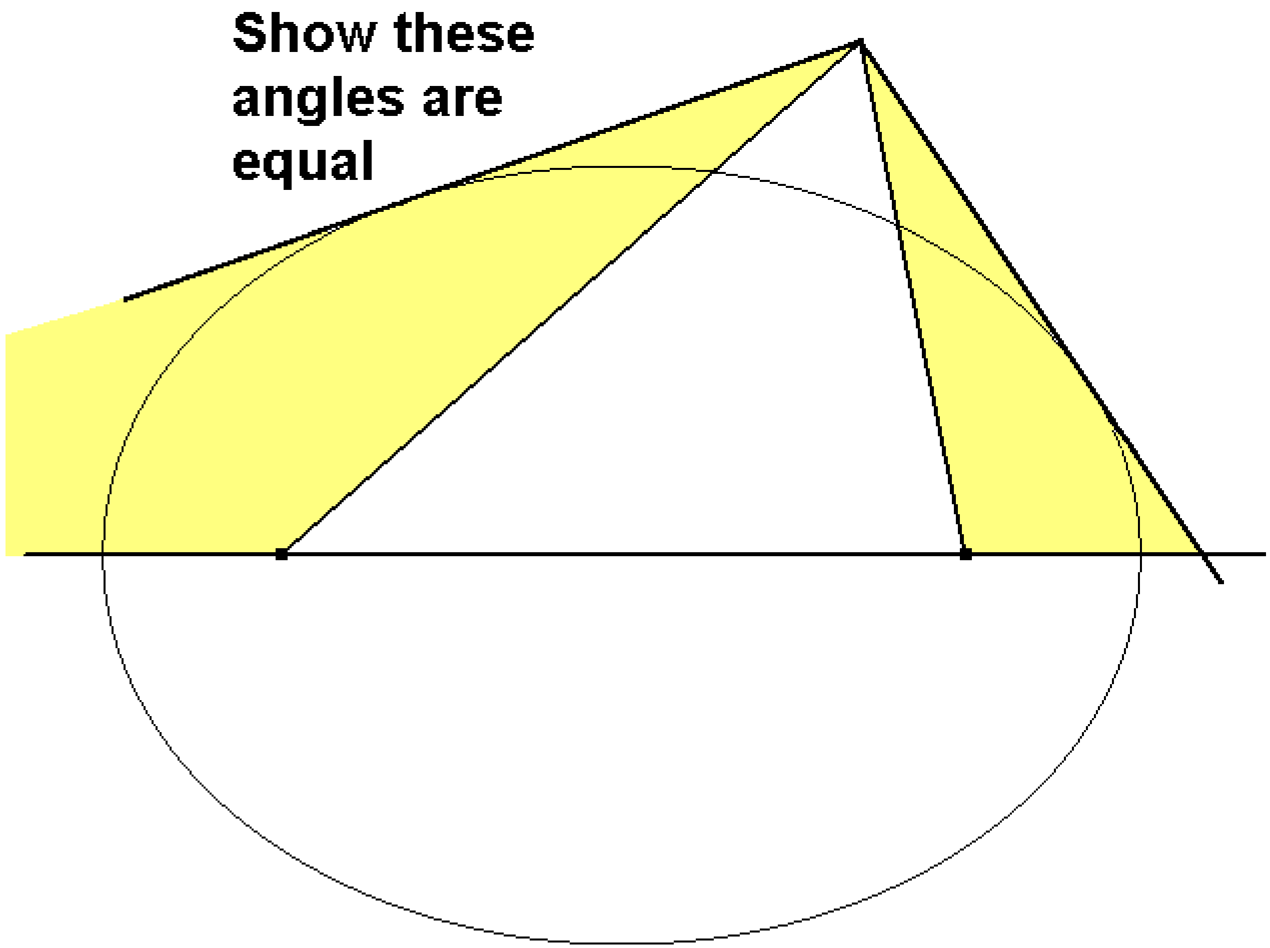
Every point of the line except *P* outside the ellipse

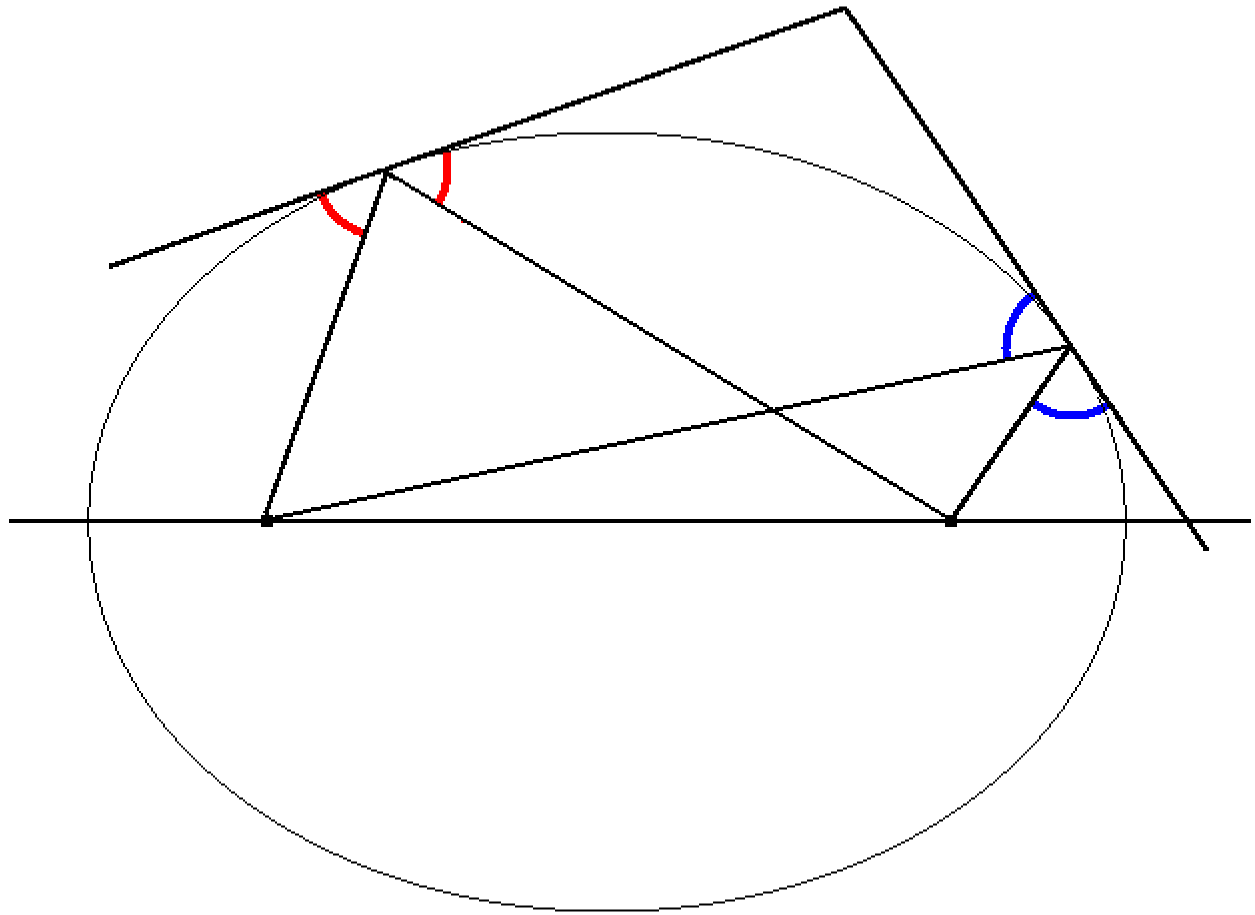
The line is tangent to the ellipse at *P*

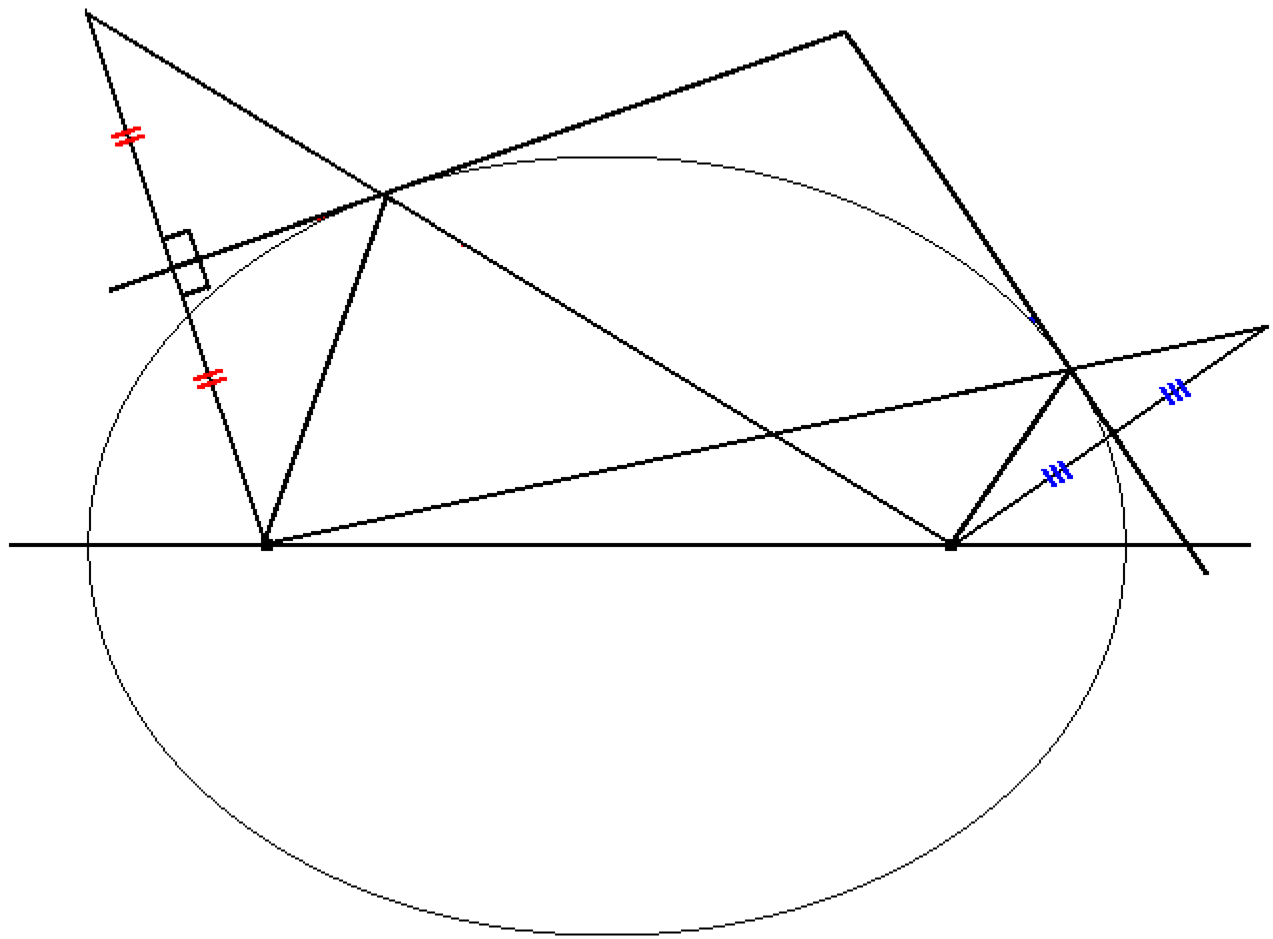
Ellipse Fact 2

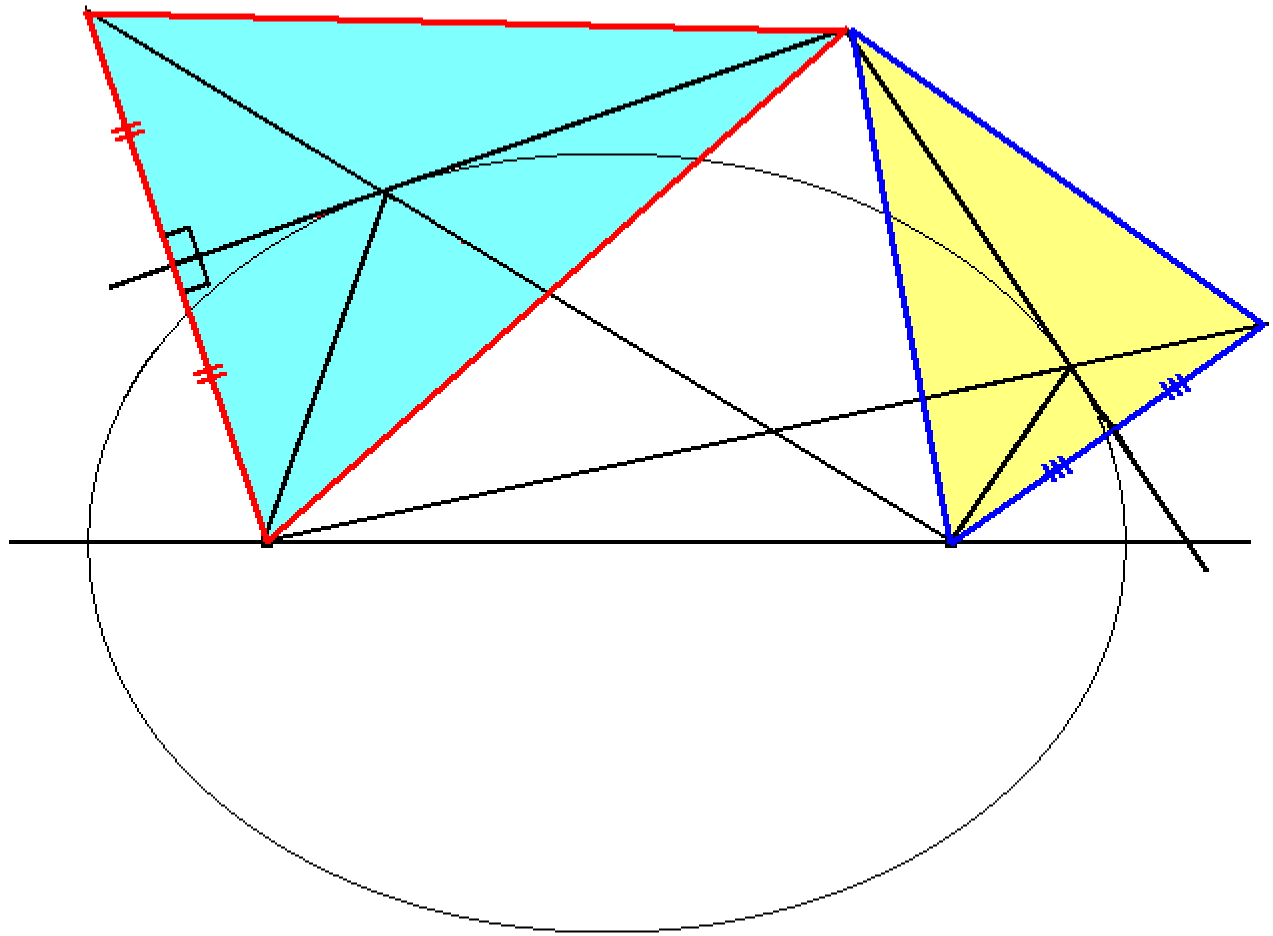
- Lines L and M from point P are both tangent to an ellipse with foci E and F
- *Fact*: Lines EP and FP make equal angles with lines L and M
- If and only if: Given one tangent line equal angles implies tangency of other line

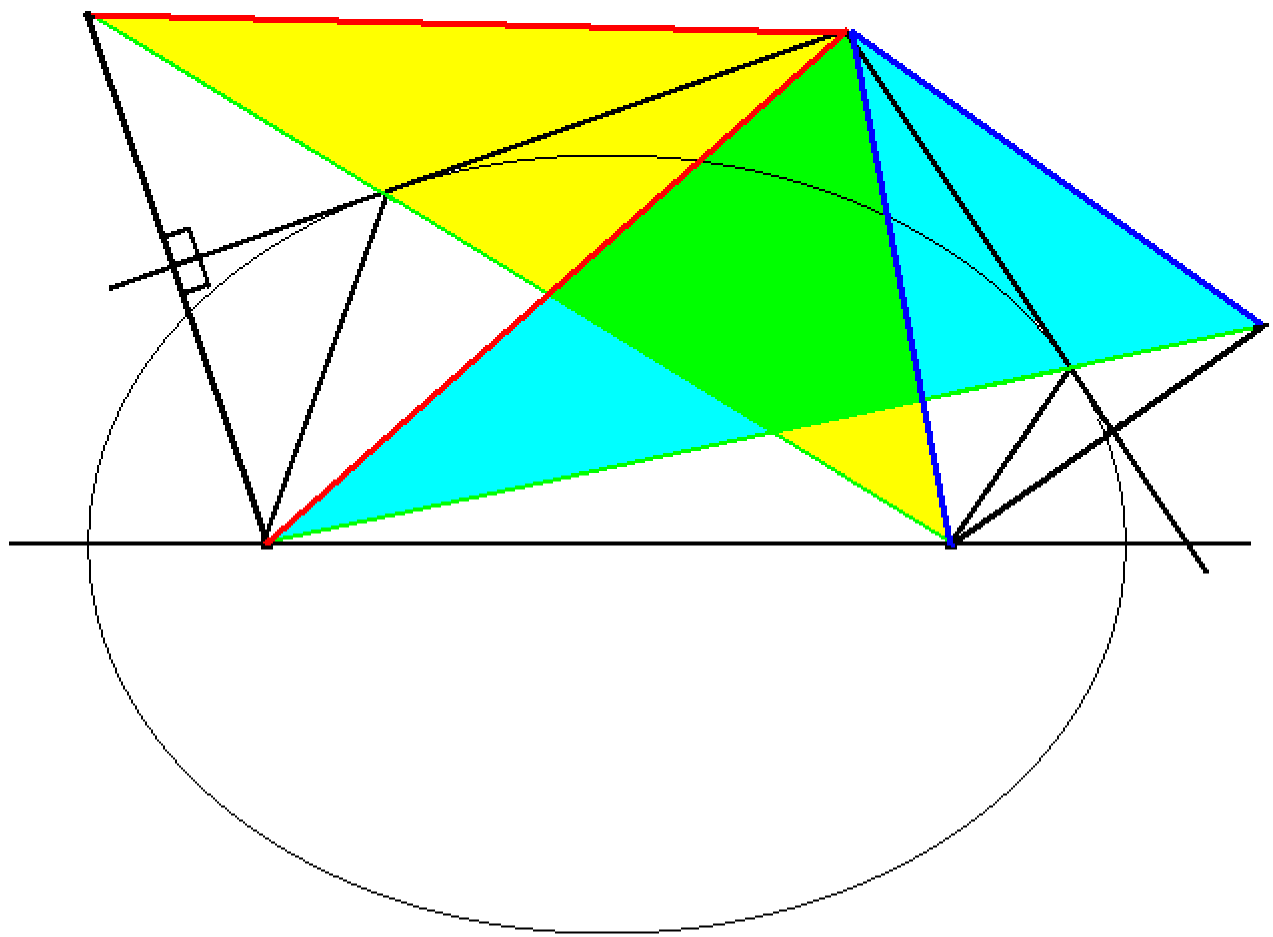


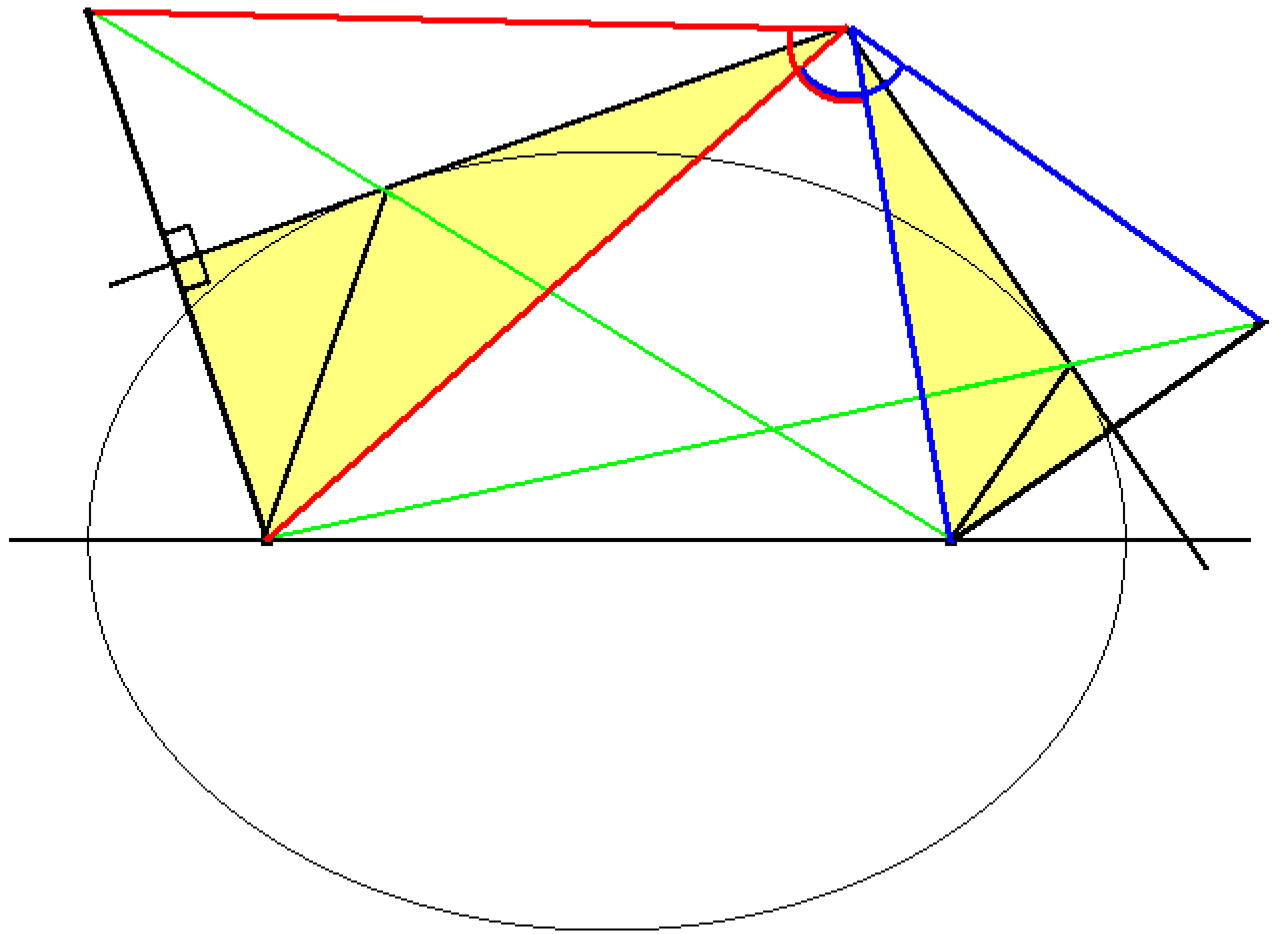

Show these
angles are
equal











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Proof Overview

- Notation: $p(z)$ has roots z_1, z_2, z_3
  $p'(z)$ has roots z_4, z_5;
- Moving triangles around
- An ellipse with foci z_4, z_5 and going through the midpoint of a side of the triangle, is actually tangent there
- The ellipses defined separately for the three sides of the triangle are actually all the same ellipse

Moving Triangles Around

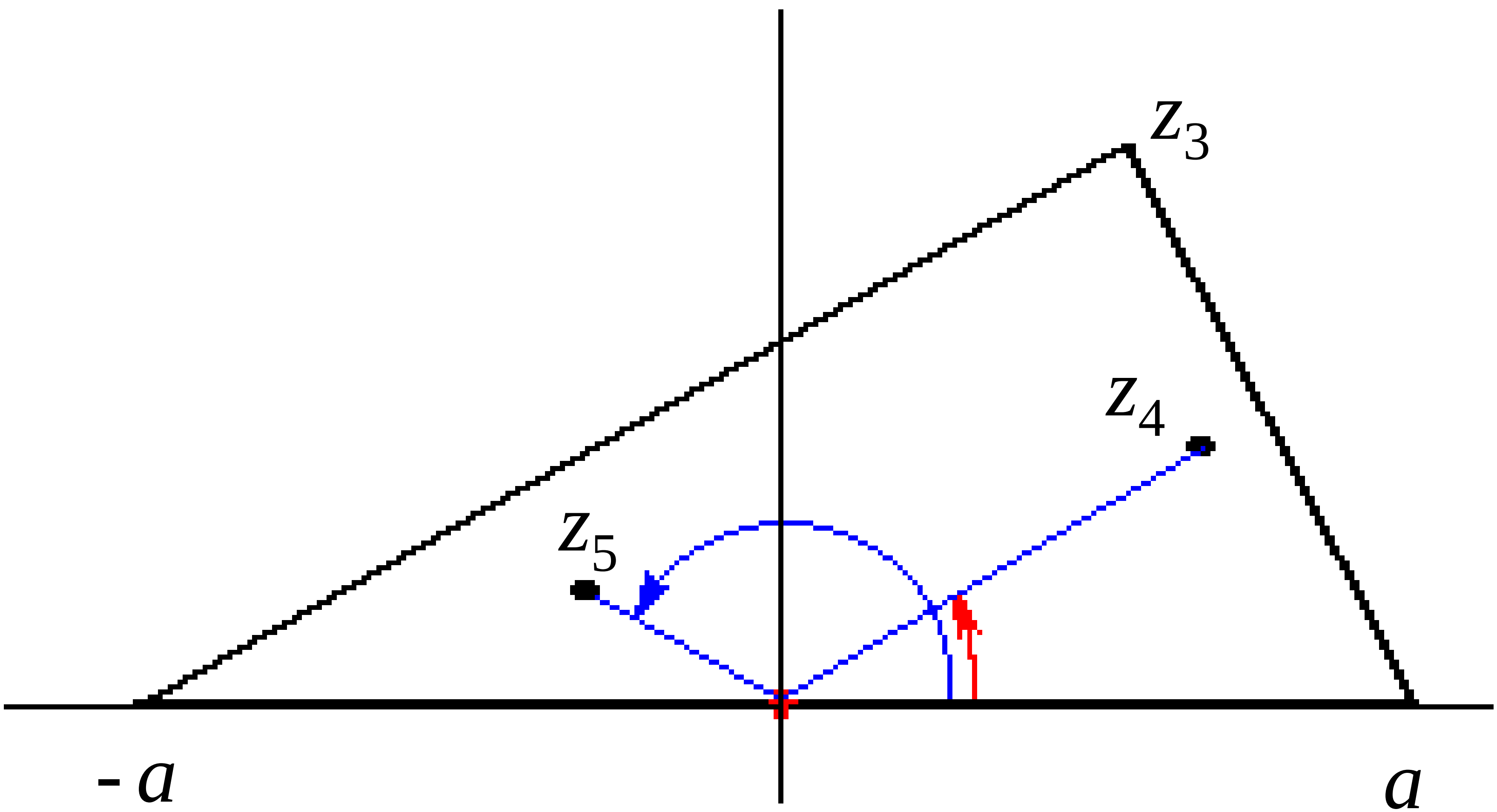
- We can rotate and translate a triangle without changing the geometry of the inscribed ellipse
- Rotation and translation is imposed by a linear function

  $z \rightarrow \alpha z + \beta$ α, β complex constants, $|\alpha| = 1$
- Carry three vertices of a triangle to new locations
- Each triangle defines a cubic polynomial
- Same function carries the roots of the derivative for the original triangle to roots of the derivative for the new triangle
- We can put the triangle in any location and with any orientation before proving the theorem

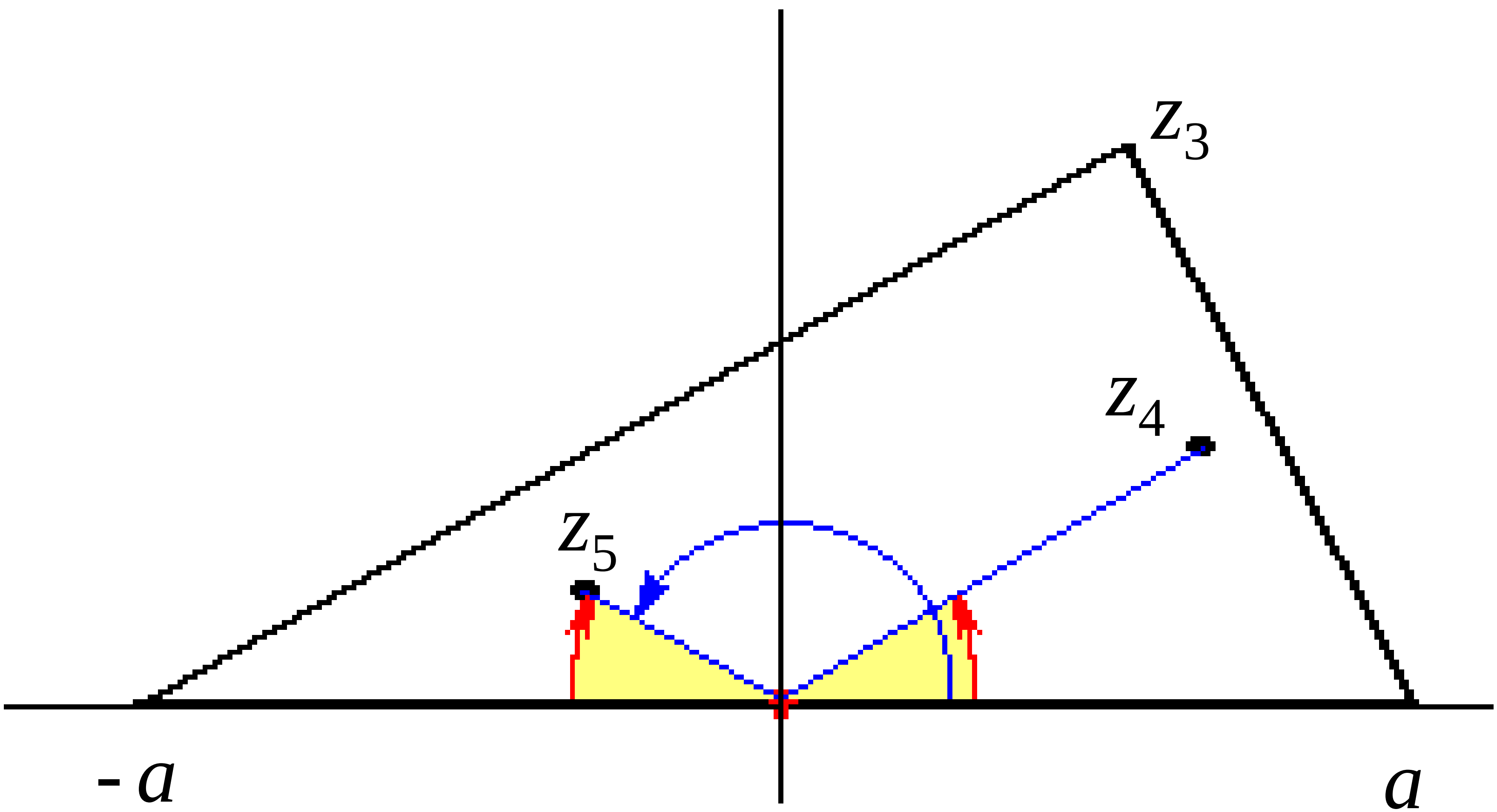
Ellipse Through One Midpoint

- Position triangle so that its base is on the *x* axis, with midpoint at the origin, and with the rest of the triangle in the upper half plane
- $z_1 = -a$, $z_2 = a$, with a real.
- $p(z) = (z+a)(z-a)(z- z_3) = (z^2-a^2)(z- z_3)$
  $= z^3 - z_3z^2 - a^2z + a^2z_3$
- $p'(z) = 3z^2 - 2z_3z - a^2$
- Roots of this quadratic are z_4 and z_5
- $z_4 z_5 = - a^2 / 3$ is negative real, angle 180°
- The angles for z_4 and z_5 add up to 180°

Angles add to 180°

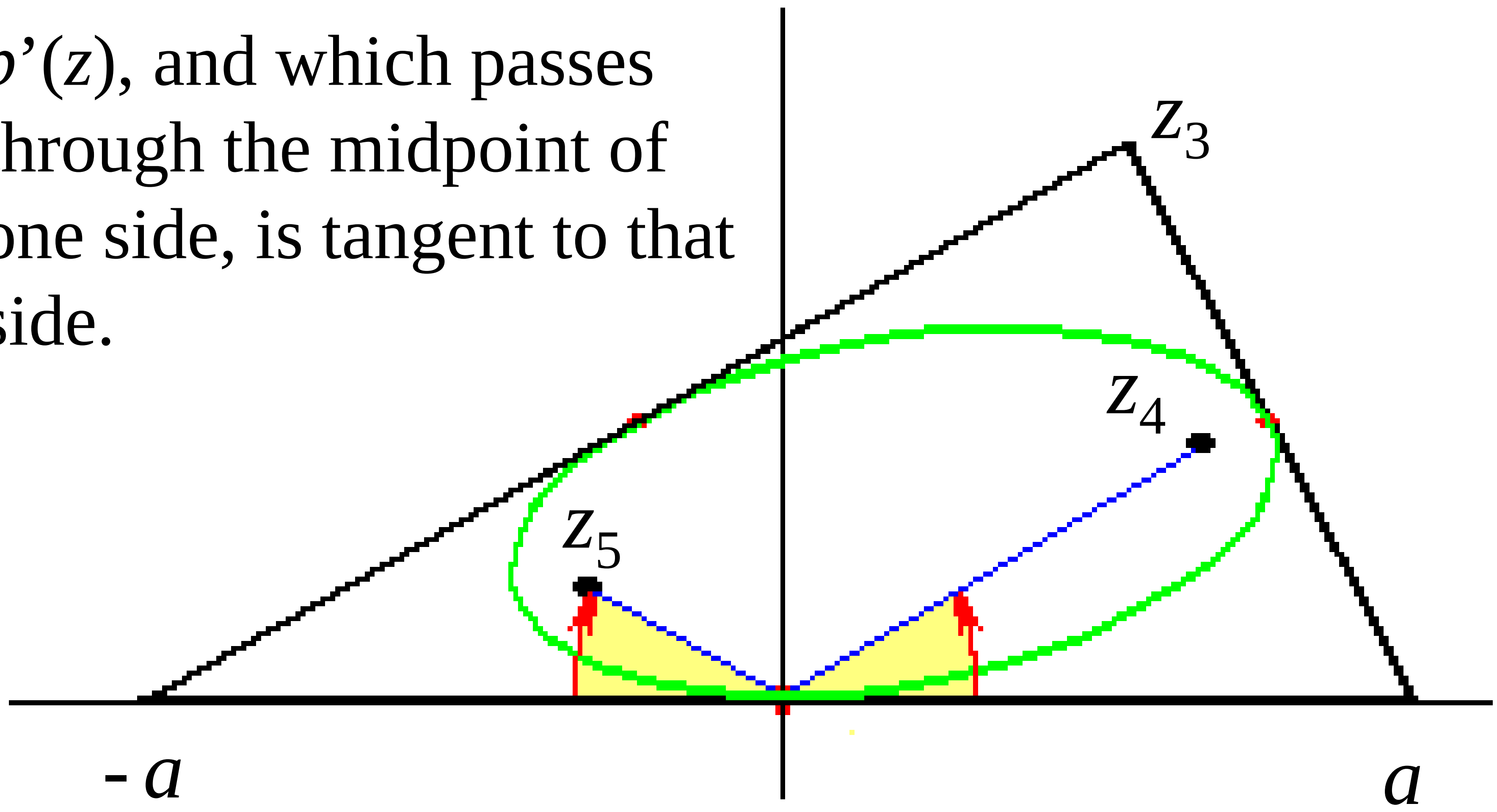


RED Angles are EQUAL



Ellipse is Tangent to *x* Axis

Conclusion: an ellipse with foci at the roots of $p'(z)$, and which passes through the midpoint of one side, is tangent to that side.



Final Step

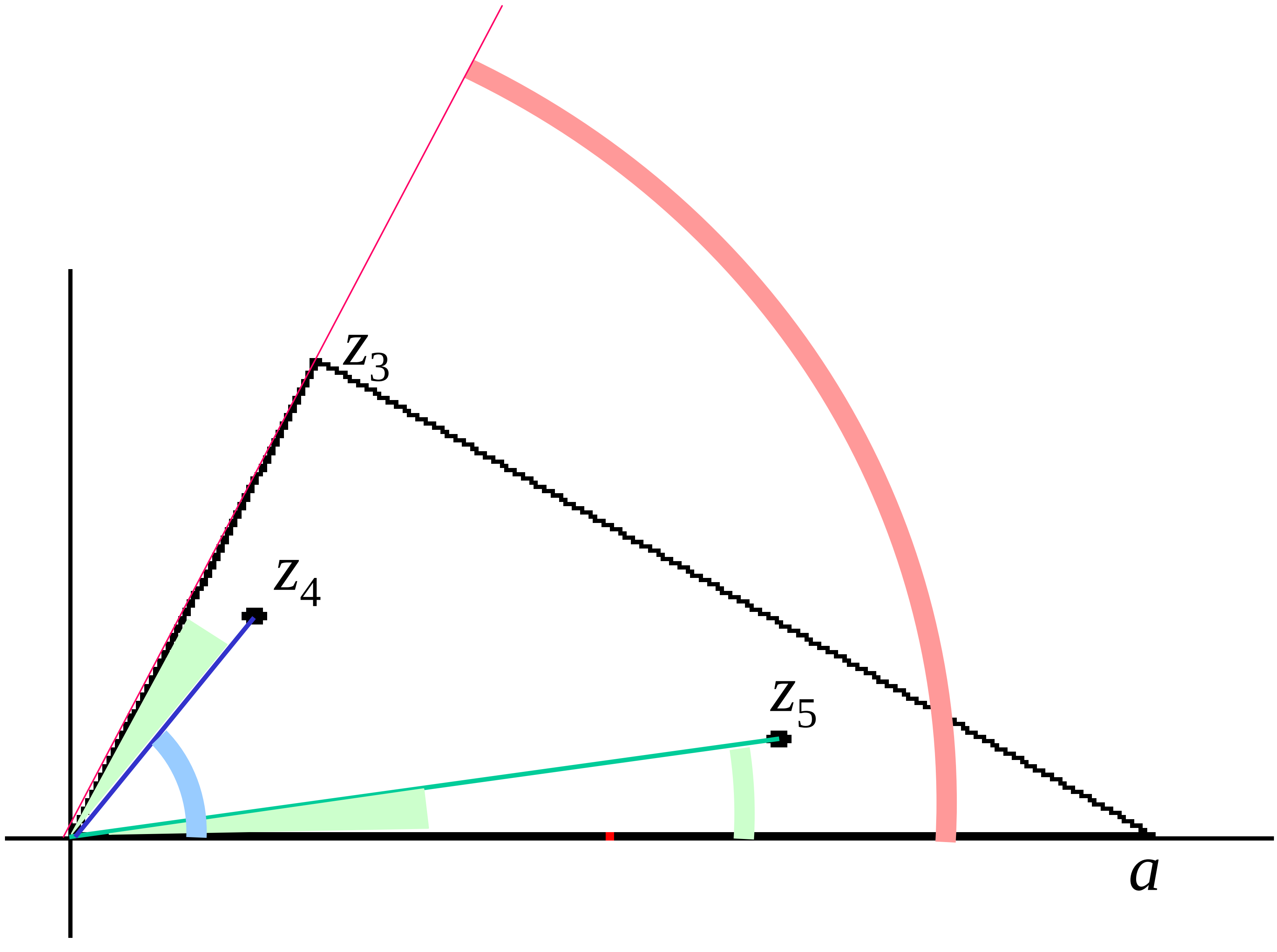
- Use the roots of $p'(z)$ as foci
- Make three ellipses; one for the midpoint of each side of the triangle
- We know that each ellipse is actually tangent to the corresponding side at the midpoint
- Now we show that the three ellipses are all the same
- Needed result: The ellipse with foci at the roots of $p'(z)$ and tangent to one side at its midpoint is also tangent to the other two sides

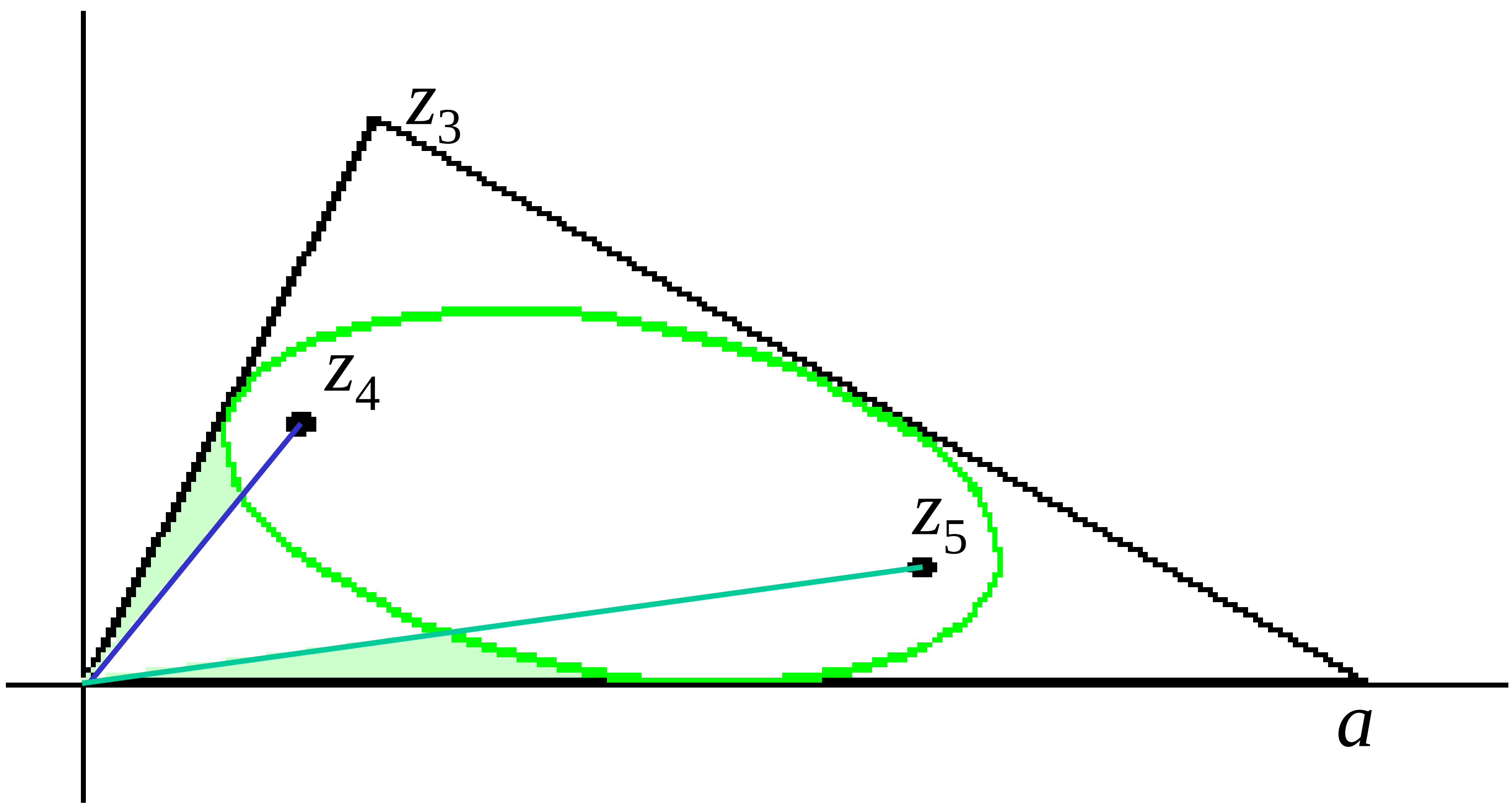
Final Step (continued)

- Needed result: The ellipse tangent to one side at its midpoint is also tangent to the other two sides
- Enough to show it is tangent to one other side
- Recall, with given foci, there can be only one ellipse tangent to a given side
- Let E_1 and E_2 be the ellipses tangent at the midpoints of the sides 1 and 2, respectively
- Suppose E_1 is also tangent to side 2
- Then E_1 and E_2 have the same foci, and are both tangent to the same side, so are identical

Final Step (continued)

- Position the triangle with one vertex at the origin, 2nd vertex at $a > 0$ on the x axis, and 3rd vertex at z_3 in upper half plane
- $p(z) = z(z - a)(z – z_3) = z^3 – (a + z_3) z^2 + az_3 z$
- $p'(z) = 3z^2 – 2(a + z_3) z + az_3$
- Roots of this quadratic are z_4 and z_5
- $z_4 z_5 = (a/3)z_3$ on same line as z_3
- Conclusion: angle for z_4 plus angle for z_5 = angle for z_3


z_3
z_4
z_5
a



z_3
z_4
z_5
a

QED

- That completes the proof
- Thanks and Finis