Math Modeling in the Prior-to-Calculus Curriculum

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Critique of College Algebra

- Too abstract
- Lacks intrinsic significance
- Lacks overall theme or story
- May carry symbolic manipulation a bit too far

One Response: Use modeling as a vehicle for algebra/precalc instructional goals.

Goals for Modeling Approach

- Retain main college algebra / precalc goals RE elementary functions and algebraic skills
- Intrinsic value/interest/significance
- Coherent story line
- Show power of algebra in context
- General education perspective on how math actually gets applied
- Decreased emphasis on abstract manipulative skills
- Highlight college algebra topics most likely to appear in client discipline introductory courses

Persistent Themes

- Discrete Sequential Data: $a_1, a_2, a_3, \cdots$ and approximating models; applying math through models
- Recursive patterns are easy to formulate: dependence of a_{n+1} on a_n. Difference equations
- Solutions to difference equations: explicit equation for a_n as a function $f(n)$; extension to continuous models
- Parameterized *families* of difference equations and solutions; fitting a model to actual data by choosing *best* values for parameters
- Direct prediction: evaluate $f(n)$ to predict data value number n
- Inverse Prediction: invert $f(n)$ to predict for which n the data value will reach a specified value
- Graphical, Numerical, and Theoretical methods

Analysis Procedures

- Numerical experimentation, exploration
  Algebra: Express general relationships
  Properties of model (function) families

- Fitting a model to actual data by choosing *best* values for parameters
  Algebra: Express problem, parameters

- Direct Prediction
  Algebra: Algebraic simplification

- Inverse Prediction
  Algebra: Solve equations

Geometric Growth

- Each term is a fixed multiple of the preceding term; equivalently, each term increases by a constant percentage over the preceding term
- Applications: population growth, compound interest, radioactive decay, drug elimination/metabolization, passive cooling/heating
- Example: population doubles each year (increases by 100%)
  Difference equation $p_{n+1} = 2p_n$
  Solution: $p_n = p_0 2^n$
  Typical questions: What will the population be in year n? When will population reach 60000?

Algebra Skills

- Properties of exponential functions Ab^t
- Graphs
- Significance of parameters A, b
- Solving equations; logarithms

Mixed Growth

- Each term combines a fixed multiple of the preceding term with a fixed increment;
- Applications: amortized loans, installment savings, repeated drug doses, chemical reactions, pollution
- Difference equation: $a_{n+1} = ra_n + d$
- Solutions are shifted exponentials: $Ar^t + C$
- Horizontal asymptote = equilibrium value $(r < 1)$

Example

- Example: Pollution flows out of a lake in proportion to the existing concentration, but flows into the lake at a constant absolute rate
- Difference equation $p_{n+1} = .9p_n + 3$ (one tenth of the pollution flows out, and three more units are added, each unit of time)
- Solution: $p_n = p_0(.9^n) + 3(1 - .9^n)/(1 - .9)$
  $= (p_0 - 30).9^n + 30$
- Typical questions: What will the pollution load be in year n? When will it reach 100? What will happen in the long term?

Finding the Solution

Numerical pattern:

$$
\begin{aligned}
p_0 &= 20 \\
p_1 &= 20(.9) + 3 \\
p_2 &= 20(.9^2) + 3(.9 + 1) \\
p_3 &= 20(.9^3) + 3(.9^2 + .9 + 1) \\
&\vdots \\
p_8 &= 20(.9^8) + 3(.9^7 + .9^6 + \cdots + .9 + 1) \\
p_n &= 20(.9^n) + 3(.9^{n-1} + .9^{n-2} + \cdots + .9 + 1)
\end{aligned}
$$

Algebraic Simplification

- Solution of difference equation is naturally derived in this form:

$$p_n = 20(.9^n) + 3\left(\frac{.9^n - 1}{.9 - 1}\right)$$

- More convenient equivalent form:

$$p_n = (p_0 - 30).9^n + 30$$

- This shows an important use of algebra: transforming symbolic expressions
- Here, it appears in a natural context generally absent in algebra classes

Equilibrium and Fixed Point

- Traditional Asymptote formulation:

$$\lim_{t \to \infty} f(t) = C$$

- Difference equation formulation (fixed point):

$$p_{n+1} = p_n$$

- Example:

$$\begin{aligned} p_{n+1} &= p_n \\ .9p_n + 3 &= p_n \\ p_n &= 30 \end{aligned}$$

Algebra Skills

- Properties of shifted exponentials $Ab^t + C$
- Graphs, horizontal asymptotes
- Significance of parameters A, b
- Solving equations; logarithms
- Finding fixed points
- Deriving general form of solution
- Transforming Expressions